Luis Adauto Medeiros
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Introduction to Exact Control Theory Method Hum

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INTRODUCTION TO EXACT CONTROL THEORY. METHOD HUM

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Preface

This book presents some results concerning existence of solutions and exact controllability for the wave equation in domains cylindrical and non cylindrical. There is one chapter dedicated to the Timoshenko system.

It is opportune to register that it is employed in the Pos Graduate Courses of " Instituto de Matemática - UFRJ."

We acknowledge Professor Ivo Fernandez Lopes for reading the previous manuscrypt and by his constructive remarks. We also register our thanks the " Editor da Universidade Estadual da Paraíba," for the inclusion of this book in the collection of its publications.

Campina Grande - PB, October, 2013

The Authors

Introduction

This book is part of lectures given by one of the authors in 1992/93 on Partial Differential Equations at Instituto de Matemática, UFRJ, Rio de Janeiro, RJ.

In order to fix the notation and terminology we will do a brief introduction to the spaces $W^{m,p}(\Omega)$. For a complete information of these subjects, the reader can look Lions [32], Medeiros-Rivera [50].

In the study of strong solution, Section I, we used general methods which could be applied even in the non linear case. However in the linear case using eigenvectors we obtain an easier proof.

Let us represent by Ω a bounded open set of $\mathbb{R}^n$ with boundary Γ. By Q we represent the cylinder $\Omega\times]0,T[$, $T>0$ real number. For $1\leq p<+\infty$, we denote by $L^p(\Omega)$ the space of real functions v measurable in Ω such that the power p, i.e. $|v|^p$, is Lebesgue integrable in Ω. This is a Banach space with the norm

$$||v||^p_{L^p(\Omega)} = \int_\Omega |v(x)|^p\,dx.$$

When $p=\infty$, $L^\infty(\Omega)$ means the space of all essentially bounded real functions in Ω, with the norm:

$$||v||_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |v(x)|.$$

We prove that $L^\infty(\Omega)$ is a Banach space.

When $p=2$ we have a Hilbert space $L^2(\Omega)$ with the inner product

$$(u,v) = \int_\Omega u(x)v(x)\,dx,$$

and induced norm

$$|v|^2 = \int_\Omega |v(x)|^2\,dx.$$

By C_0^∞ we represent the space of real function defined in Ω, infinitely differentiable and with compact support in Ω. By $\mathcal{D}(\Omega)$ we represent the space of $C_0^\infty(\Omega)$ with the notion of convergence: φ_n and all its derivatives converge uniformly to φ and its derivatives in K. A distribution on Ω, as defined by Laurent Schwartz, is a continuous linear form T on $\mathcal{D}(\Omega)$. Its derivative of order α, $D^\alpha T$, is defined, for each α, by

$$\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha \varphi\rangle,$$

for all $\varphi \in \mathcal{D}(\Omega)$. Note that $\langle T, \varphi\rangle$ is the evaluation of T in φ, i.e. $T(\varphi)$.

By $W^{m,p}(\Omega)$ we represent the Sobolev spaces of order m, that is, the space of all real functions $v \in L^p(\Omega)$ such that $D^\alpha v \in L^p(\Omega)$ for all $|\alpha| \leq m$. In $W^{m,p}(\Omega)$ we define the norm:

$$||v||_{m,p}^p = \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha v(x)|^p \, dx.$$

It follows that $W^{m,p}(\Omega)$ with this norm is a Banach space. By $W_0^{m,p}(\Omega)$ we represent the closure of $\mathcal{D}(\Omega)$ in $W^{m,p}(\Omega)$.

When $p = 2$, the space $W^{m,2}(\Omega)$ is represented by $H^m(\Omega)$, which is a Hilbert space with the inner product

$$(u, v) = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u(x) \cdot D^\alpha v(x) \, dx$$

and norm:

$$|v|_{m,2}^2 = \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha v(x)|^2 \, dx.$$

In particular, we use, frequently, in this book, the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$. We have

$$H^1(\Omega) = \left\{v \in L^2(\Omega); \frac{\partial v}{\partial x_i} \in L^2(\Omega),\, i = 1, 2, \ldots, n\right\}$$

with the inner product

$$((u, v)) = \int_\Omega u(x)v(x)\, dx + \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx,$$

and norm:

$$||v||^2 = \int_\Omega |v(x)|^2 \, dx + \int_\Omega |\nabla v(x)|^2 \, dx.$$

By ∇ we represent the gradient operator. In $H_0^1(\Omega)$ we obtain an equivalent norm given by

$$||v||^2 = \int_\Omega |\nabla v(x)|^2 \, dx.$$

Let us consider the Laplace operator Δ defined by the triplet $\{H_0^1(\Omega), L^2(\Omega); ((\cdot,\cdot))\}$. Its domain is, for regular Γ,

$$D(-\Delta) = \{v \in L^2(\Omega); \Delta v \in L^2(\Omega)\} = H_0^1(\Omega) \cap H^2(\Omega).$$

When Γ is of class C^2 we prove that the norm $H^2(\Omega)$ defined in $H_0^1(\Omega)\cap H^2(\Omega)$ is equivalent to the norm

$$|v|_\Delta^2 = \int_\Omega |\Delta v(x)|^2\, dx,$$

that is, the norm defined by the Laplace's operator. By this reason, we consider $H_0^1(\Omega)\cap H^2(\Omega)$ with the norm $|v|_\Delta$.

Given a Banach space X and a real number $T > 0$, we represent by $L^p(0,T;X)$, with $1 \leq p < \infty$, the space of vector functions $v\colon\,]0,T[\to X$, measurable and such that $||v(t)||_X^p$ is integrable in $]0,T[$. In $L^p(0,T;X)$ we define the norm:

$$||v||_{L^p(0,T;X)}^p = \int_0^T ||v(t)||_X^p\, dt.$$

As in numerical case we define $L^\infty(0,T;X)$ with the norm:

$$||v||_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{0<t<T} ||v(t)||_X\, .$$

We prove that $L^p(0,T;X)$, $1 \leq p \leq \infty$ are Banach's spaces.

Note that we represent the inner product and norm, respectively, in $L^2(\Omega)$ and $H_0^1(\Omega)$ by the notations: $(\cdot,\cdot)$; $|\cdot|$; $((\cdot,\cdot))$ and $||\cdot||$.

We also appreciate the suggestions of Ricardo Fuentes about Chapter 8. To Wilson Góes my thanks for the beautiful work of TEX.

Epigraph

"... la troisième, de conduire par ordre mes pensées, en commençant par les objects les plus simples, et les plus aisés a connaitre, pour monter peu à peu, come par degrés, jusqu'á à la connaissance des plus composés..."

René Descartes – Discours de la Méthode

Chapter 1

Strong Solutions

1.1 Strong Solutions

This section is dedicated to solve the following boundary value problem:
Given

$$\phi^0 \in H_0^1(\Omega) \cap H^2(\Omega); \quad \phi^1 \in H_0^1(\Omega) \text{ and } f \in L^1(0,T;H_0^1(\Omega)),$$

find a numerical function $u\colon Q \to \mathbb{R}$ satisfying the conditions:

$$\left| \begin{array}{l} \phi'' - \Delta\phi = f \quad \text{a.e. in} \quad Q, \\ \phi = 0 \quad \text{on} \quad \Sigma, \\ \phi(x,0) = \phi^0(x), \phi'(x,0) = \phi^1(x) \quad \text{on} \quad \Omega. \end{array} \right. \tag{*}$$

Note that ϕ' is $\dfrac{\partial \phi}{\partial t}$ and $\phi(t)$ is the function $\phi(t)\colon x \to \phi(x,t)$. Consequently $\phi(x,0)$ can be written $\phi(0)$. Thus the initial data is $\phi(0) = \phi^0$ and $\phi'(0) = \phi^1$. The following theorem solve the problem.

Theorem 1.1 (Existence and Uniqueness) *If $\phi^0 \in H_0^1(\Omega) \cap H^2(\Omega)$; $\phi^1 \in H_0^1(\Omega)$ and $f \in L^1(0,T;H_0^1(\Omega))$, there exists only one function $\phi\colon Q \to \mathbb{R}$ such that:*

$$\phi \in L^\infty(0,T;H_0^1(\Omega) \cap H^2(\Omega)) \tag{1.1}$$

$$\phi' \in L^\infty(0,T;H_0^1(\Omega)) \tag{1.2}$$

$$\phi'' \in L^1(0,T;L^2(\Omega)) \tag{1.3}$$

$$\phi'' - \Delta\phi = f \quad \textit{a.e. in} \quad Q \tag{1.4}$$

$$\phi(0) = \phi^0 \quad \textit{and} \quad \phi'(0) = \phi^1 \tag{1.5}$$

Proof: Let $(w_\nu)_{\nu\in\mathbb{N}}$, $(\lambda_\nu)_{\nu\in\mathbb{N}}$ be, respectively, the eigenfunctions and the eigenvalues of the spectral problem

$$((w_j, v)) = \lambda_j (w_j, v) \quad \text{for all} \quad v \in H_0^1(\Omega).$$

Approximated Problem. Let us consider the m-dimensional subspace of $H_0^1(\Omega) \cap H^2(\Omega)$ denoted by $V_m = [w_1, w_2, \ldots, w_m]$, generated by the m-first eigenfunctions w_ν, $\nu = 1, 2, \ldots, m, \ldots$. Then, we propose the approximated problem:

$$\left|\begin{array}{l} \text{Find } \phi_m(t) \in V_m\,, \text{ such that:} \\ (\phi_m''(t), v) + ((\phi_m(t), v)) = (f(t), v) \text{ for all } v \in V_m\,. \\ \phi_m(0) = \phi_m^0\,, \text{ converges to } \phi^0 \text{ in } H_0^1(\Omega) \cap H^2(\Omega). \\ \phi_m'(0) = \phi_m^1 \text{ converges to } \phi^1 \text{ in } H_0^1(\Omega). \end{array}\right. \tag{1.6}$$

Remark 1.1 *Observe that if $\phi_m(t) \in V_m$ then*

$$\phi_m(t) = \sum_{i=1}^{m} g_i(t) w_i\,, \tag{1.7}$$

where $g_i(t)$ for $1 \le i \le m$, are determined by the equations $(1.6)_2$. When we substitute $\phi_m(t)$, given by (1.7), in $(1.6)_2$ we obtain, for $v = w_j$, $1 \le j \le m$,

$$g_j''(t) + \lambda_j g_j(t) = (f, w_j), \quad 1 \le i \le m, \tag{1.8}$$

which is a system of m ordinary differential equations of second order with constants coefficients λ_j.

The initial conditions for (1.8) are obtained by the conditions $(1.6)_3$ and $(1.6)_4$. We consider $H_0^1(\Omega) \cap H^2(\Omega)$ with the equivalent norm defined by the Laplace operator, since Γ is regular. The approximations for ϕ^0 and ϕ^1 are:

$$\phi_m^0 = \sum_{i=1}^{m} (\phi^0, w_i) w_i \quad \textit{and} \quad \phi_m^1 = \sum_{i=1}^{m} (\phi^1, w_i) w_i\,. \tag{1.9}$$

Then the initial conditions for (1.8) are:

$$g_j(0) = (\phi^0, w_j) \quad \textit{and} \quad g_j'(0) = (\phi^1, w_j). \tag{1.10}$$

The system (1.8) with initial conditions (1.10) has only one solution defined in $[0, T]$. Consequently the system (1.6) has solution $\phi_m(t)$ defined in $[0, T]$. In the next step we obtain a priori estimates.

First a priori estimate. Consider $v = 2\phi'_m(t)$ in $(1.6)_2$. We obtain:

$$\frac{d}{dt}\left(|\phi'_m(t)|^2 + ||\phi_m(t)||^2\right) = 2(f(t), \phi'_m(t)).$$

Integrating from 0 to $t \leq T$, we get:

$$|\phi'_m(t)|^2 + ||\phi_m(t)||^2 \leq |\phi^1_m|^2 + ||\phi^0_m|^2 + \int_0^T |f(s)|\, ds + \int_0^t |f(s)|\, |\phi'_m(s)|^2\, ds.$$

By Gronwall's inequality, it follows:

$$|\phi'_m(t)|^2 + ||\phi_m(t)||^2 < C_1 \quad \text{for} \quad 0 \leq t \leq T. \tag{1.11}$$

Second a priori estimate. By the choice of $(w_\nu)_{\nu\in\mathbb{N}}$, it follows that $-\Delta\phi'_m(t) \in V_m$. Then it is correct to take $v = -2\Delta\phi'_m(t)$ in $(1.6)_2$, obtaining:

$$\frac{d}{dt}\left(|\nabla\phi'_m(t)|^2 + |\Delta\phi_m(t)|^2\right) = 2(\nabla f(t), \nabla\phi'_m(t)).$$

Integrating this equality from 0 to $t \leq T$, we obtain:

$$\begin{aligned} |\nabla\phi'_m(t)|^2 + |\Delta\phi_m(t)|^2 &\leq |\nabla\phi^1_m|^2 + |\Delta\phi^0_m|^2 + \\ &+ \int_0^T |\nabla f(s)|\, ds + \int_0^t |\nabla f(s)|\, |\nabla\phi'_m(s)|^2\, ds. \end{aligned}$$

By hypothesis $|\nabla f(s)| \in L^1(0,T)$, then by Gronwall's inequality applied to the last inequality we obtain:

$$|\nabla\phi'_m(t)|^2 + |\Delta\phi_m(t)|^2 < C_2\,, \quad \text{for} \quad 0 \leq t \leq T. \tag{1.12}$$

Then from (1.11) and (1.12) we obtain:

$$\Delta\phi_m \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)) \tag{1.13}$$

$$\phi'_m \quad \text{is bounded in} \quad L^\infty(0,T;H^1_0(\Omega)) \tag{1.14}$$

We extract a subsequence $(\phi_\mu)_{\mu\in\mathbb{N}}$ of $(\phi_m)_{m\in\mathbb{N}}$, such that:

$$\Delta\phi_m \rightharpoonup \xi = \Delta\phi \quad \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)) \tag{1.15}$$

$$\phi'_\mu \rightharpoonup \phi' \quad \text{weak star in} \quad L^\infty(0,T;H^1_0(\Omega)) \text{ and } L^\infty(0,T;L^2(\Omega)). \tag{1.16}$$

Remark 1.2 *The first estimate gives $\phi_\mu \rightharpoonup \phi$ weak star in $L^\infty(0,T;H^1_0(\Omega))$, then in the sense of distribution on Q. Therefore, $\Delta\phi_m \to \Delta\phi$ in the sense of distributions on Q. By (1.15) we obtain $\Delta\phi_\mu \to \xi$ in the sense of distributions on Q, then $\xi = \Delta\phi$.* ■

By (1.16) we obtain $(\phi''_\mu(t), v) \rightharpoonup \dfrac{d}{dt}(\phi'(t), v)$ in $\mathcal{D}'(0,T)$ for all $v \in L^2(\Omega)$. Then, fix m in $(1.6)_2$ and consider the sequence ϕ_μ as solution of the approximated problem (1.6) and let $\mu \to \infty$. We obtain:

$$\frac{d}{dt}(\phi'(t), v) - (\Delta\phi(t), v) = (f(t), v)$$

in the sense of $\mathcal{D}'(0,T)$, for all $v \in V_m$. By density it is true for all $v \in H_0^1(\Omega) \cap H^2(\Omega)$. In particular, for all $v \in \mathcal{D}(\Omega)$. Then we have:

$$-\int_0^T\int_\Omega \phi'(t)v\theta'\,dxdt = \int_0^T\int_\Omega \Delta\phi(t)v\theta\,dxdt = \int_0^T\int_\Omega f(t)v\theta\,dxdt$$

for all $v \in \mathcal{D}(\Omega)$, $\theta \in \mathcal{D}(0,T)$. By density of the finite sums of products $v\theta$, $v \in \mathcal{D}(\Omega)$ and $\theta \in \mathcal{D}(0,T)$ in $\mathcal{D}(Q)$, we obtain:

$$-\int_Q \phi'\psi'\,dxdt - \int_Q \Delta\phi\psi\,dxdt = \int_Q f\psi\,dxdt$$

for all $\psi \in \mathcal{D}(Q)$. Then,

$$\langle \phi'', \psi\rangle = \int_Q (\Delta\phi + f)\psi\,dxdt,$$

and it follows that the distribution ϕ'' is defined on Q by $\Delta\phi + f \in L^1(0,T;L^2(\Omega))$. Then we identify ϕ'' to a function of $L^1(0,T;L^2(\Omega))$ and still represent this function by ϕ''. We have:

$$\int_Q (\phi'' - \Delta\phi - f)\psi\,dxdt = 0$$

for all $\psi \in \mathcal{D}(Q)$. Whence, Lemma of Du Bois Raymond implies:

$$\phi'' - \Delta\phi = f \quad \text{a.e. in} \quad Q.$$

Then we prove (1.1), (1.2) and (1.3) of the Theorem 1.1. ■

To prove uniqueness, let ϕ, $\widehat{\phi}$ be two solutions in the conditions of the Theorem 1.1. It follows that $\zeta = \phi - \widehat{\phi}$ is solution of $\zeta'' - \Delta\zeta = 0$ a.e. in Q, $\zeta(0) = 0$ and $\zeta'(0) = 0$. Since, by (1.16), $\zeta' \in L^\infty(0,T;H_0^1(\Omega))$ make sense the integrals

$$\int_\Omega \zeta''\zeta'\,dx - \int_\Omega \Delta\zeta\zeta'\,dx = 0.$$

Whence,

$$\frac{d}{dt}(|\zeta'(t)|^2 + ||\zeta(t)||^2) = 0,$$

what implies, $\zeta = 0$ on Q. ■

The solution ϕ obtained in Theorem 1.1 is called **strong solution** of the mixed problem (*), or for the linear wave equation.

Theorem 1.2 (Energy Inequality) *If ϕ is strong solution, then we have the energy inequality*

$$|\nabla\phi'(t)|^2 + |\Delta\phi(t)|^2 \le |\nabla\phi^1|^2 + |\Delta\phi^0|^2 + 2\int_0^t (\nabla f(s), \nabla\phi'(s))\,ds. \tag{1.17}$$

Proof: Taking $v = -\Delta\phi'_m(t) \in V_m$ in the approximated equations $(1.6)_2$, we obtain:

$$|\nabla\phi'_m(t)|^2 + |\Delta\phi_m(t)|^2 = |\nabla\phi^1_m|^2 + |\Delta\phi^0_m|^2 + 2\int_0^t (\nabla f(s), \nabla\phi'_m(s))\,ds. \tag{1.18}$$

Let $\theta > 0$ be an step function on $]0,T[$. In (1.18) take $m = \mu$, multiply both sides by θ and integrate on $[0,T]$. We obtain:

$$\begin{aligned} \int_0^T |\nabla\phi'_\mu(t)|^2\theta(t)\,dt + \int_0^T |\Delta\phi_\mu(t)|^2\theta(t)\,dt = \int_0^T |\nabla\phi^1_\mu|^2\theta(t)\,dt + \\ + \int_0^T |\Delta\phi^0_\mu|^2\theta(t)\,dt + 2\int_0^T \theta(t)\int_0^t (\nabla f(s), \nabla\phi'_\mu(s))\,dsdt. \end{aligned} \tag{1.19}$$

By the convergences (1.15), (1.16) and the lower semicontinuity of the norms with respect to the weak convergence, we obtain:

$$\int_0^T |\nabla\phi'(t)|^2\theta(t)\,dt \le \varliminf_{\mu} \int_0^T |\nabla\phi'_\mu(t)|^2\theta(t)\,dt \tag{1.20}$$

$$\int_0^T |\Delta\phi(t)|^2\theta(t)\,dt \le \varliminf_{\mu} \int_0^T |\Delta\phi_\mu(t)|^2\theta(t)\,dt \tag{1.21}$$

Taking $\varliminf_{\mu}$ in both sides of (1.19), taking in account (1.20) and (1.21) and noting that $\varliminf u + \varliminf v \le \varliminf (u+v)$, we obtain:

$$\begin{aligned} &\int_0^T |\nabla\phi'(t)|^2\theta(t)\,dt + \int_0^T |\Delta\phi(t)|^2\theta(t)\,dt \le \\ &\le \int_0^T |\nabla\phi^1|^2\theta(t)\,dt + \int_0^T |\Delta\phi^0|^2\theta(t)\,dt + \\ &+2\int_0^T \left(\int_0^t (\nabla f(s), \nabla\phi'(s))\,ds\right)\theta(t)\,dt \end{aligned} \tag{1.22}$$

Remark 1.3 *Let be $v \in L^1(0,T)$. We say that $s \in]0,T[$ is a Lebesgue point of v, if for $h > 0$ such that $]s-h,\, s+h[\subset]0,T[$ then*

$$\lim_{h\to 0} \frac{1}{2h}\int_{s-h}^{s+h} v(\xi)\,d\xi = v(s).$$

It is proved that if $v \in L^1(0,T)$, then almost all points s of $]0,T[$ are Lebesgue's points of v.

Let us return to (1.22) and observe that the functions in the integrands of (1.22) are $L^1(0,T)$. If $s \in]0,T[$, let us consider the step function $\theta_h(t) = \theta(t)$ on $]s-h,\ s+h[\subset]0,T[$ and zero in the complement. Then θ_h is permissible in (1.22). Substituting θ by θ_h in (1.22), dividing both sides by $2h$ and letting $h \to 0$ we obtain, for $t \in [0,T]$, because $\theta > 0$:

$$|\nabla \phi'(t)|^2 + |\Delta \phi(t)|^2 \leq |\nabla \phi^1|^2 + |\Delta \phi^0|^2 + 2\int_0^t (\nabla f(s), \nabla \phi'(s))\, ds \tag{1.23}$$

a.e. in $[0,T]$. ■

Before to prove another form of energy inequality (1.23) we prove a Gronwall's inequality, Brezis [4] or Gomes [17].

Lemma 1.1 *Let $m \in L^1(0,T,\mathbb{R})$ such that $m \geq 0$ a.e. in $]0,T[$ and $a \geq 0$ real constant. Suppose $g \in L^\infty(0,T)$, $g \geq 0$ on $]0,T[$ verifying the inequality:*

$$\frac{1}{2}\, g(t)^2 \leq 2a^2 + 2\int_0^t m(s)g(s)\, ds$$

for all $t \in]0,T[$. Then:

$$g(t) \leq 2\left(a + \int_0^t m(s)\, ds\right) \quad in \quad [0,T].$$

Proof: For $\varepsilon > 0$ let us consider the function $\psi_\varepsilon > 0$ in $[0,T]$ defined by:

$$\psi_\varepsilon(t) = 2(a+\varepsilon)^2 + 2\int_0^t m(s)g(s)\, ds.$$

Whence,

$$\frac{d}{dt}\, \psi_\varepsilon(t) = 2m(t)g(t).$$

We have $\frac{1}{2}\, g^2 \leq \psi_\varepsilon$ or $g(t) \leq \sqrt{2}\, \sqrt{\psi_\varepsilon(t)}$. Since ψ_ε is absolutely continuous and $\psi_\varepsilon(t) \geq 2\varepsilon^2$, we have

$$\frac{d}{dt}\, \psi_\varepsilon(t)^{1/2} = \frac{1}{2\psi_\varepsilon(t)^{1/2}}\, \frac{d\psi_\varepsilon(t)}{dt} \leq \sqrt{2}\, m(t).$$

Integrating this inequality from 0 to t, we have:

$$\psi_\varepsilon(t)^{1/2} \leq \psi_\varepsilon(0)^{1/2} + \sqrt{2}\int_0^t m(s)\, ds \quad \text{for all} \quad t \in [0,T].$$

Since $g(t) \leq \sqrt{2}\, \psi_\varepsilon(t)^{1/2}$, $0 \leq t \leq T$, we obtain, from the above inequality, after $\varepsilon \to 0$,

$$g(t) \leq 2\left(a + \int_0^t m(s)\, ds\right) \quad \text{in} \quad [0,T]. \tag{1.24}$$

■

Corollary 1.1 *If ϕ is the strong solution of Theorem 1.1, we have the inequality:*

$$|\nabla\phi'(t)| + |\Delta\phi(t)| \leq C\left(|\nabla\phi^1| + |\Delta\phi^0| + \int_0^t |\nabla f(s)|\, ds\right) \tag{1.25}$$

in $[0,T]$.

Proof: In fact from (1.23) we obtain

$$(|\nabla\phi'(t)| + |\Delta\phi(t)|)^2 \leq 2(|\nabla\phi^1| + |\Delta\phi^0|)^2 + 4\int_0^t |\nabla f(s)|\, |\nabla\phi'(s)| ds.$$

If we define:

$$g(t) = |\nabla\phi'(t)| + |\Delta\phi(t)|,$$

we obtain from the above inequality:

$$\frac{1}{2}\, g(t)^2 \leq 2a^2 + 2\int_0^t m(s)g(s)\, ds,$$

where

$$\alpha = |\nabla\phi^1| + |\Delta\phi^0|.$$

By Lemma 1.1 we obtain (1.25). ■

Note that if the boundary Γ of Ω is C^2, then the norm of $H_0^1(\Omega)\cap H^2(\Omega)$ and that one given by the Laplace operator are equivalents, as we already seen in Introduction.

We then obtain from (1.25) the inequality:

$$\begin{aligned} &||\phi'||_{L^\infty(0,T;H_0^1(\Omega))} + ||\phi||_{L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))} \leq \\ &\leq C\big(||\phi^1||_{H_0^1(\Omega)} + ||\phi^0||)_{H_0^1(\Omega)\cap H^2(\Omega)} + ||f||_{L^1(0,T;H_0^1(\Omega))}\big). \end{aligned} \tag{1.26}$$

■

Theorem 1.3 (Regularity) *The strong solution $\phi = \phi(x,t)$ has the regularity:*

$$\phi \in C^0([0,T]; H_0^1(\Omega)\cap H^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)). \tag{1.27}$$

Proof: The strong solution ϕ, which exists by Theorem 1.1, is weak limit of a sequence of approximations of the type:

$$\phi_m(t) = \sum_{i=1}^m g_i(t)w_i \tag{1.28}$$

with $g_i(t)$, $1 \leq i \leq m$, solutions of the following system of ordinary differential equations:

$$g_j''(t) + \lambda_j g_j(t) = (f, w_j), \quad 1 \leq j \leq m, \tag{1.29}$$

plus initial conditions:

$$g_j(0) = (\phi^0, w_j) \quad \text{and} \quad g_j'(0) = (\phi^1, w_j). \tag{1.30}$$

The solution of this initial value problem is given, Lagrange's method of variation of constants, by:

$$g_j(t) = (\phi^0, w_j) \cos \sqrt{\lambda_j} t + \frac{1}{\sqrt{\lambda_j}} (\phi^1, w_j) \sin \sqrt{\lambda_j} t +$$
$$+ \frac{1}{\sqrt{\lambda_j}} \int_0^t (f(s), w_j) \sin \sqrt{\lambda_j}\,(t-s)\, ds,$$

$1 \leq j \leq m$.

Whence, the approximated solution is given by:

$$\begin{aligned} \phi_m(x,t) = \sum_{i=1}^{m} \Bigg[&(\phi^0, w_i) \cos \sqrt{\lambda_i} t + \frac{1}{\sqrt{\lambda_i}} (\phi^1, w_i) \sin \sqrt{\lambda_i} t + \\ &+ \frac{1}{\sqrt{\lambda_i}} \int_0^t (f(s), w_i) \sin \sqrt{\lambda_i}\,(t-s)\, ds \Bigg] w_i \,. \end{aligned} \tag{1.31}$$

In the proof we suppose f regular to use Parseval identity. The general case $L^1(0,T;H_0^1(\Omega))$ we approximated by regular case.

Step 1. $\phi \in C^0([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$.

In fact, it is sufficient to prove that $(\phi_m)_{m\in\mathbb{N}}$ is a Cauchy's sequence in $C^0([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$. Note that we consider in $H_0^1(\Omega) \cap H^2(\Omega)$ the norm defined by the Laplace operator. In fact, let us consider $m, n \in \mathbb{N}$ and suppose $m > n$. We have:

$$||\phi_m(t) - \phi_n(t)||_V^2 = \left\| \sum_{i=n+1}^{m} g_i(t) w_i \right\|_V^2 = \left| \sum_{i=n+1}^{m} g_i(t) \Delta w_i \right|_{L^2(\Omega)}^2 .$$

Noting that $-\Delta w_i = \lambda_i w_i$, we obtain by Pithagoras' theorem:

$$||\phi_m(t) - \phi_n(t)||_V^2 = \sum_{i=n+1}^{m} |g_i(t)\lambda_i|_{\mathbb{R}}^2.$$

We have:

$$\begin{aligned} |g_i(t)\lambda_i|_{\mathbb{R}}^2 &= |(\phi^0, w_i)\lambda_i \cos\sqrt{\lambda_i} t + \frac{1}{\sqrt{\lambda_i}} (\phi^1, w_i)\lambda_i \sin\sqrt{\lambda_i} t + \\ &+ \frac{1}{\sqrt{\lambda_i}} \int_0^t (f(s), w_i)\lambda_i \sin\sqrt{\lambda_i}\,(t-s)\, ds|_{\mathbb{R}}^2 \le \\ &\le \left\{ |(\phi^0, w_i)\lambda_i|_{\mathbb{R}} + \left| \frac{1}{\sqrt{\lambda_i}} (\phi^1, w_i)\lambda_i \right|_{\mathbb{R}} + \frac{1}{\sqrt{\lambda_i}} \int_0^t |(f(s), \lambda_i w_i)|_{\mathbb{R}}\, ds \right\}^2 . \end{aligned}$$

Applying twice the elementary inequality $(a+b)^2 \le 2a^2 + 2b^2$

$$\begin{aligned} |g_i(t)\lambda_i|_{\mathbb{R}}^2 &\le 4|(\Delta\phi^0, w_i)|_{\mathbb{R}}^2 + 4\left|(\phi^1, w_i)\tfrac{\lambda_i}{\sqrt{\lambda_i}}\right|_{\mathbb{R}}^2 + \\ &+2\left(\int_0^T \left|(f(s), w_i)\frac{\lambda_i}{\sqrt{\lambda_i}}\right| ds\right)^2 . \end{aligned} \tag{1.32}$$

Note that $\phi^0 = \sum_{i=1}^{\infty} \left(\left(\phi^0, \frac{w_i}{\lambda_i}\right)\right)_V \frac{w_i}{\lambda_i}$ and by Pithagoras' theorem

$$||\phi^0||_V^2 = \sum_{i=1}^{\infty} \left| \left(\left(\phi^0, \frac{w_i}{\lambda_i}\right)\right)_V \right|_{\mathbb{R}}^2 .$$

We know that $\left(\left(\phi^0, \frac{w_i}{\lambda_i}\right)\right)_V = \left(\Delta\phi^0, \Delta\frac{w_i}{\lambda_i}\right) = -(\Delta\phi^0, w_i)$. Whence

$$\sum_{i=n+1}^{m} |(\Delta\phi^0, w_i)|^2 \quad \text{converges to zero when} \quad m, n \to \infty. \tag{1.33}$$

For the second term of the right hand side of (1.32) we obtain:

$$\phi^1 = \sum_{i=1}^{\infty} \left(\left(\phi^1, \frac{w_i}{\sqrt{\lambda_i}}\right)\right) \frac{w_i}{\sqrt{\lambda_i}} ,$$

noting that $((\ ,\ ))$ is the inner product in $H_0^1(\Omega)$. Then,

$$||\phi^1||^2 = \sum_{i=1}^{\infty} \left| \left(\left(\phi^1, \frac{w_i}{\sqrt{\lambda_i}} \right) \right) \right|_{\mathbb{R}}^2$$

we have,

$$\left(\left(\phi^1, \frac{w_i}{\sqrt{\lambda_i}} \right) \right) = \left(\nabla\phi^1, \nabla \frac{w_i}{\sqrt{\lambda_i}} \right) = -(\phi^1, w_i) \frac{\lambda_i}{\sqrt{\lambda_i}}$$

whence,

$$\sum_{i=n+1}^{m} \left| (\phi^1, w_i) \frac{\lambda_i}{\sqrt{\lambda_i}} \right|_{\mathbb{R}}^2 \quad \text{converges to zero when} \quad m, n \to \infty. \tag{1.34}$$

We know that $f(s) \in H_0^1(\Omega)$, then:

$$f(s) = \sum_{i=1}^{\infty} \left(\left(f(s), \frac{w_i}{\sqrt{\lambda_i}} \right) \right) \frac{w_i}{\sqrt{\lambda_i}} \cdot$$

It follows that:

$$||f(s)||^2 = \sum_{i=1}^{\infty} \left| (f(s), w_i) \frac{\lambda_i}{\sqrt{\lambda_i}} \right|_{\mathbb{R}}^2 \cdot$$

Then, by Schwarz's inequality:

$$\left(\int_0^T \left| (f(s), w_i) \frac{\lambda_i}{\sqrt{\lambda_i}} \right| ds \right)^2 \leq T \int_0^T \left| (f(s), w_i) \frac{w_i}{\sqrt{\lambda_i}} \right|_{\mathbb{R}}^2 ds.$$

Therefore, for the last term of the right hand side of (1.32), we have:

$$\sum_{i=n+1}^{m} \left\{ \int_0^T \left| (f(s), w_i) \frac{\lambda_i}{\sqrt{\lambda_i}} \right| ds \right\}^2 \leq T \int_0^T \sum_{i=n+1}^{m} \left| (f(s), w_i) \frac{\lambda_i}{\sqrt{\lambda_i}} \right|^2 ds, \tag{1.35}$$

which converges to zero when $m, n \to \infty$, independent of t in $[0, T]$.

By (1.33), (1.34) and (1.35) we have, from (1.32):

$$\sum_{i=n+1}^{m} \left| g_i(t) \lambda_i \right|_{\mathbb{R}}^2 \quad \text{converges to zero when} \quad m, n \to \infty.$$

Consequently, the sequence $(\phi_m(t))_{m\in\mathbb{N}}$ is such that $\max_{0\leq t\leq T} ||\phi_m(t) - \phi_n(t)||_V$ converges to zero when $m, n \to \infty$ or $(\phi_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$, then convergent and its limit ϕ, which is the strong solution, belongs to $C^0([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$. ■

Step 2. $\phi^1 \in C^0([0,T]; H_0^1(\Omega))$.

The method is the same of Step 1. First we take the derivative with respect to t of the approximated solution and obtain:

$$\phi_m'(x,t) = \sum_{i=1}^{m} g_i'(t) w_i \,,$$

where

$$g_i'(t) = -\,(\phi^0, w_i)\sqrt{\lambda_i}\sin\sqrt{\lambda_i}t + (\phi^1, w_i)\cos\sqrt{\lambda_i}t + \\ + \int_0^t (f(s), w_i)\cos\sqrt{\lambda_i}(t-s)\,ds.$$

We need to prove that $(\phi_m')_{m\in\mathbb{N}}$ is a Cauchy sequence in $C^0([0,T]; H_0^1(\Omega))$. Suppose $m > n$, $m, n \in \mathbb{N}$. We have:

$$||\phi_m'(t) - \phi_n'(t)||^2 = \left\| \sum_{i=n+1}^{m} g_i'(t) w_i \right\|^2 = \left| \sum_{i=n+1}^{m} g_i'(t) \nabla w_i \right|^2_{L^2(\Omega)}.$$

By Pithagoras' theorem, we have:

$$||\phi_m'(t) - \phi_n'(t)||^2 = \sum_{i=n+1}^{m} \left| g_i'(t)\sqrt{\lambda_i} \right|^2_{\mathbb{R}}.$$

We have,

$$\left| g_i'(t)\sqrt{\lambda_i} \right|^2_{\mathbb{R}} \le 4\left|(\phi^0, w_i)\lambda_i\right|^2_{\mathbb{R}} + 4\left|(\phi^1, w_i)\sqrt{\lambda_i}\right|^2_{\mathbb{R}} + \\ + 2\left\{ \int_0^t (f(s), w_i)\sqrt{\lambda_i}\,ds \right\}^2. \tag{1.36}$$

Note that

$$(\phi^0, w_i)\lambda_i = (\Delta\phi^0, w_i); \quad (\phi^1(w_i)\sqrt{\lambda_i} = (\phi^1, w_i)\frac{\lambda_i}{\sqrt{\lambda_i}}$$

and $(f(s), w_i)\sqrt{\lambda_i} = (f(s), w_i)\dfrac{\lambda_i}{\sqrt{\lambda_i}}\cdot$

Therefore, by the same argument used to obtain (1.33), (1.34) and (1.35) we have that $(\phi_m')_{m\in\mathbb{N}}$ is Cauchy's sequence in $C^0([0,T]; H_0^1(\Omega))$ and it follows that $\phi' \in C^0([0,T]; H_0^1(\Omega))$.

■

Chapter 2

Weak Solutions

2.1 Weak Solutions

We consider now the mixed problem of Chapter 1, but under weak hypotheses on the initial conditions ϕ^0, ϕ^1.

Theorem 2.1 *Consider*

$$\phi^0 \in H_0^1(\Omega),\ \ \phi^1 \in L^2(\Omega) \quad \textit{and} \quad f \in L^1(0,T;L^2(\Omega)). \tag{2.1}$$

There exists only one function $\phi\colon Q \to \mathbb{R}$ *satisfying the conditions:*

$$\phi \in L^\infty(0,T;H_0^1(\Omega)) \tag{2.2}$$

$$\phi' \in L^\infty(0,T;L^2(\Omega)) \tag{2.3}$$

$$\frac{d}{dt}\,(\phi'(t),v) + ((\phi(t),v)) = (f(t),v) \tag{2.4}$$

in the sense of $\mathcal{D}'(0,T)$*, for all* $v \in H_0^1(\Omega)$

$$\phi'' \in L^1(0,T;H^{-1}(\Omega)) \ \textit{and}\ \phi'' - \Delta\phi = f \ \textit{in}\ L^1(0,T;H^{-1}(\Omega)) \tag{2.5}$$

$$\phi(0) = \phi^0, \quad \phi'(0) = \phi^1. \tag{2.6}$$

The function ϕ obtained by Theorem 2.1 is called **weak solution** of the mixed problem (*).

Proof: We prove this theorem approximating the weak solutions by a sequence of strong solutions. In fact, let us consider the approximations of ϕ^0, ϕ^1 and f

$$\left|\begin{array}{l}\phi_m^0 \in H_0^1(\Omega)\cap H^2(\Omega) \text{ such that } \phi_m^0 \to \phi^0 \text{ in } H_0^1(\Omega),\\ \phi_m^1 \in H_0^1(\Omega) \text{ such that } \phi_m^1 \to \phi^1 \text{ in } L^2(\Omega),\\ f_m \in C^0([0,T];C^1(\overline{\Omega})) \text{ such that} f_m \to f \text{ in } L^1(0,T;L^2(\Omega)).\end{array}\right. \tag{2.7}$$

Taking ϕ_m^0 , ϕ_m^1 and f_m as data, Theorem 1.1 of Chapter 1 says that there exists only one function $\phi_m \colon Q \to \mathbb{R}$ satisfying the conditions:

$$\left|\begin{array}{l} \phi_m \in L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega)), \\ \phi_m' \in L^\infty(0,T;H_0^1(\Omega)), \\ \phi_m'' \in L^\infty(0,T;L^2(\Omega)), \\ (\phi_m''(t),v) + ((\phi_m(t),v)) = (f_m(t),v), \quad \text{in} \quad]0,T[\,, \\ \text{for all} \quad v \in L^2(0,T;H_0^1(\Omega)), \\ \phi_m(0) = \phi_m^0\,, \quad \phi_m'(0) = \phi_m^1. \end{array}\right. \tag{2.8}$$

The next step consists in obtaining precise estimates for ϕ_m , given by (2.8), such that the limit is the solution claimed in Theorem 2.1.

Taking $v = \phi_m'(t)$ in $(2.8)_4$, we obtain:

$$\frac{d}{dt}\,(|\phi_m'(t)|^2 + ||\phi_m(t)||^2) = 2(f(t),\phi_m'(t))$$

or

$$|\phi_m'(t)|^2 + ||\phi_m(t)||^2 \le |\phi_m^1|^2 + ||\phi_m^0||^2 + \int_0^T |f_m(t)|\,dt + \int_0^t |f_m(s)|\,|\phi_m'(s)|^2\,ds.$$

By the convergences (2.7), we get from the above inequality:

$$|\phi_m'(t)|^2 + ||\phi_m(t)||^2 \le K + \int_0^t |f_m(s)|\,|\phi_m'(s)|^2\,ds$$

for all $t \in [0,T]$. By Gronwall inequality it implies:

$$|\phi_m'(t)|^2 + ||\phi_m(t)||^2 < C, \quad \text{for all} \quad t \in [0,T]. \tag{2.9}$$

From (2.9) follows the existence of a subsequence $(\phi_n)_{n\in\mathbb{N}}$ such that:

$$\left|\begin{array}{lll} \phi_n & \text{converges to } \phi \text{ weak star in} & L^\infty(0,T;H_0^1(\Omega)) \\ \phi_n' & \text{converges to } \phi' \text{ weak star in} & L^\infty(0,T;L^2(\Omega)) \end{array}\right. \tag{2.10}$$

By $(2.8)_4$ we have:

$$\frac{d}{dt}\,(\phi_n'(t),v) + ((\phi_n(t),v)) = (f_n(t),v) \tag{2.11}$$

for all $v \in H_0^1(\Omega)$. Multiplying both sides of (2.11) by $\theta \in \mathcal{D}(0,T)$ and integrating by parts, we obtain:

$$-\int_0^T (\phi_n'(t),v)\theta'(t)\,dt + \int_0^T ((\phi_n(t),v))\theta(t)\,dt = \int_0^T (f_n(t),v)\theta(t)\,dt \tag{2.12}$$

for all $v \in H_0^1(\Omega)$. Taking the limit in (2.13) when $n \to \infty$, taking in account (2.10) and $(2.7)_3$, we obtain a function $\phi\colon Q \to \mathbb{R}$ such that:

$$\left| \begin{array}{l} \phi' \in L^\infty(0,T; H_0^1(\Omega)), \\ \phi' \in L^\infty(0,T; L^2(\Omega)), \\ \dfrac{d}{dt}\,(\phi'(t), v) + ((\phi(t), v)) = (f(t), v) \\ \text{in } \mathcal{D}'(0,T), \text{ for all } v \in H_0^1(\Omega). \end{array} \right. \tag{2.13}$$

We will prove that $\phi'' \in L^1(0,T; H^{-1}(\Omega))$. In fact, from (2.12), when n goes to infinity, we obtain:

$$-\int_0^T (\phi'(t), v)\theta'(t)\,dt + \int_0^T \langle -\Delta\phi(t), v\rangle \theta(t)\,dt = \int_0^T (f(t), v)\theta(t)\,dt \tag{2.14}$$

for all $v \in H_0^1(\Omega)$ and $\theta \in \mathcal{D}(0,T)$. Then, defining $g(t) = f(t) - \Delta\phi(t) \in H^{-1}(\Omega)$, we obtain from (2.14):

$$-\int_0^T \phi'(t)\theta'(t)\,dt = \int_0^T g(t)\theta(t)\,dt. \tag{2.15}$$

By (2.2) and (2.3) it follows that $\phi', g \in L^1(0,T; H^{-1}(\Omega))$ and satisfies (2.15). Then by Temam [66] Lemma 1.1 it follows that:

$$\phi'(t) = \xi + \int_0^t g(s)\,ds, \quad \xi \in H^{-1}(\Omega) \quad \text{constant.} \tag{2.16}$$

Whence

$$\phi' \in C^0([0,T]; H^{-1}(\Omega)). \tag{2.17}$$

By (2.16) we obtain:

$$\langle \phi'', \theta\rangle = \langle g, \theta\rangle \quad \text{for all} \quad \theta \in \mathcal{D}(0,T),$$

what implies:

$$\phi'' \in L^1(0,T; H^{-1}(\Omega))$$

and $\phi'' = g$ in $L^1(0,T; H^{-1}(\Omega))$ that is,

$$\phi'' - \Delta\phi = f \quad \text{in} \quad L^1(0,T; H^{-1}(\Omega)). \tag{2.18}$$

■

To complete the proof we need to verify the initial conditions and the uniqueness. First, let us prove that $u(0) = u_0$. In fact, we have:

$$\phi_m(t) = \phi_m(0) + \int_0^t \phi_m'(s)\,ds. \tag{2.19}$$

Taking norm in $L^2(\Omega)$ of both sides in (2.19), we obtain:

$$\phi_m \quad \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \tag{2.20}$$

Then, there exists a subsequence $(\phi_\nu)_{\nu\in\mathbb{N}}$ such that:

$$\int_0^T (\phi_\nu(t),v)\theta'(t)\,dt \quad \text{converges to} \quad \int_0^T (\phi(t),v)\theta'(t)\,dt \tag{2.21}$$

for all $\theta \in C^1([0,T])$ such that $\theta(0)=1$ and $\theta(T)=0$. From $(2.10)_2$ we have:

$$\int_0^T (\phi'_\nu(t),v)\theta\,dt \quad \text{converges to} \quad \int_0^T (\phi'(t),v)\theta\,dt \tag{2.22}$$

for $\theta \in C^1([0,T])$, $\theta(0)=1$ and $\theta(T)=0$.

By (2.21) and (2.22) we obtain:

$$\int_0^T \frac{d}{dt}[(\phi_\nu(t),v)\theta]dt \quad \text{converges to} \quad \int_0^T \frac{d}{dt}[(\phi(t),v)\theta]dt$$

or $(\phi_\nu(0),v)$ converges to $(\phi(0),v)$. Note that $\phi(0)$ make sense. We know that $(\phi_\nu(0),v)$ converges to (ϕ^0,v) for all $v\in H_0^1(\Omega)$. Then $\phi(0)=\phi^0$. ■

We prove now that $\phi'(0)=\phi^1$. In fact, let be $\delta>0$ and consider the function θ_δ defined by:

$$\theta_\delta(t) = \left| \begin{array}{l} -\dfrac{t}{\delta}+1 \quad \text{if} \quad 0\le t\le \delta \\ 0 \quad \text{if} \quad \delta < t \le T \end{array} \right.$$

which belongs to $H^1(0,T)$. Multiplying both sides of the approximated equation $(2.8)_4$ by $\theta_\delta(t)$ and integrating by parts we obtain:

$$\begin{aligned} &-(\phi'_\nu(0),v) + \frac{1}{\delta}\int_0^\delta (\phi'_\nu(t),v)\,dt + \int_0^\delta ((\phi_\nu(t),v))\theta_\delta\,dt = \\ &= \int_0^\delta (f(t),v)\theta_\delta\,dt, \end{aligned} \tag{2.23}$$

for the subsequence $(\phi_\nu)_{\nu\in\mathbb{N}}$ obtained from (2.20). If $\nu\to\infty$ in (2.23) we get:

$$-(\phi^1,v) + \frac{1}{\delta}\int_0^\delta (\phi'(t),v)\,dt + \int_0^\delta ((\phi(t),v))\theta_\delta\,dt = \int_0^\delta (f(t),v)\theta_\delta\,dt. \tag{2.24}$$

Letting $\delta\to 0$ in (2.24) we obtain $(\phi'(0),v)=(\phi^0,v)$ for all $v\in H_0^1(\Omega)$ or $\phi(0)=\phi^1$ in $H_0^1(\Omega)$. ■

To prove uniqueness, let be ϕ and $\widehat{\phi}$ two weak solutions given by Theorem 2.1. Then $w=\phi-\widehat{\phi}$ is weak solution of

$$\left| \begin{array}{l} w''-\Delta w = 0 \quad \text{on} \quad Q, \\ w=0 \quad \text{on} \quad \Sigma, \\ w(0)=0, \quad w'(0)=0. \end{array} \right. \tag{2.25}$$

Note that $w'' \in L^1(0,T;H^{-1}(\Omega))$ and $w' \in L^\infty(0,T;L^2(\Omega))$, what does not permit to consider $\langle w'', w'\rangle$, duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Therefore there exists a method, cf. Visik-Ladyzhenskaya [69], which consists in define a new function ψ, from w, such that $\psi \in L^\infty(0,T;H_0^1(\Omega))$ and the energy method works. In fact, for $0 < s < T$ let us define:

$$\psi(t) = \left| \begin{array}{l} -\displaystyle\int_t^s w(\sigma)\, d\sigma \quad \text{if} \quad 0 < t < s, \\ 0 \quad \text{if} \quad s \le t < T \end{array} \right.$$

where w is an weak solution of (2.25). The function $\psi \in L^2(0,T;H_0^1(\Omega))$, then makes sense:

$$\int_0^T \langle w'' - \Delta w, \psi\rangle\, dt = 0. \tag{2.26}$$

Let us consider

$$w_1(\xi) = \int_0^\xi w(\sigma)\, d\sigma.$$

Whence,

$$\psi(t) = w_1(t) - w_1(s)$$

and

$$\psi'(t) = w_1'(t) = w(t).$$

We have:

$$\int_0^s \langle w'', \psi\rangle\, d\sigma = (w'(s), \psi(s)) - (w'(0), \psi(0)) - \int_0^t (w', \psi')\, d\sigma.$$

Since $\psi(s) = w'(0) = 0$ we obtain:

$$\int_0^s \langle w'', \psi\rangle\, d\sigma = -\frac{1}{2}\, |w(s)|^2. \tag{2.27}$$

From (2.26) and (2.27) we obtain:

$$-\frac{1}{2}\, |w(s)|^2 + \int_0^s ((w, \psi))\, d\sigma = 0. \tag{2.28}$$

But, $((w, \psi)) = ((\psi', \psi)) = \dfrac{1}{2}\dfrac{d}{dt}\, ||\psi(t)||^2$. Then, from (2.28) it follows:

$$|w(s)|^2 + ||\psi(0)||^2 = 0$$

proving that $w(s) = 0$ for all $s \in [0,T]$. ■

Theorem 2.2 (Energy Inequality) *If ϕ is the weak solution obtained in Theorem 2.1, then we have the energy inequality:*

$$|\phi'(t)|^2 + ||\phi(t)||^2 \le |\phi^1|^2 + ||\phi^0||^2 + 2\int_0^t (f(s), \phi'(s))\, ds \tag{2.29}$$

a.e. in $[0,T]$.

Proof: From $(2.8)_4$ with $m = \nu$, taking $v = \phi'_\nu(t)$ we obtain:

$$|\phi'_\nu(t)|^2 + ||\phi_\nu(t)||^2 \leq |\phi^1_\nu|^2 + ||\phi^2_\nu|| + 2\int_0^t (f(s), \phi'_\nu(s))\, ds.$$

By the convergences (2.10) and the same argument used in the proof of Theorem 1.2, Chapter 1, we obtain the inequality (2.29). ■

Corollary 2.1 *If ϕ is the weak solution which exists by Theorem 2.1, we have the inequality:*

$$|\phi'(t)| + ||\phi(t)|| \leq C\left(|\phi^1| + ||\phi^0|| + \int_0^T |f(s)|\, ds\right) \tag{2.30}$$

in $[0, T]$.

Proof: The same argument used to prove Corollary 1.1, of Theorem 2.1 of Chapter 1. ■

From the Corollary 1.1, we obtain:

$$||\phi'||_{L^\infty(0,T;L^2(\Omega)} + ||\phi||_{L^\infty(0,T;H^1_0(\Omega))} \leq$$
$$\leq C\big(|\phi^1|_{L^2(\Omega)} + ||\phi^0||_{H^1_0(\Omega)} + ||f||_{L^1(0,T;L^2(\Omega))}\big)$$

■

Theorem 2.3 (Regularity of Weak Solutions) *The weak solution ϕ has the following regularity:*

$$\phi \in C^0([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega)). \tag{2.31}$$

Proof: Let $(\phi_\nu)_{\nu\in\mathbb{N}}$ be the sequence of strong solutions that approximate the weak solution ϕ. Then, if $m, n \in \mathbb{N}$, $m > n$, we have:

$$(\phi''_m(t) - \phi''_n(t), v) + ((\phi_m(t) - \phi_n(t), v)) = (f_m(t) - f_n(t), v)$$

for all $v \in L^2(0,T; H^1_0(\Omega))$, by $(2.8)_4$. Taking $v = \phi'_m(t) - \phi'_n(t)$, we obtain

$$\frac{d}{dt}\left(|\phi'_m(t) - \phi'_n(t)|^2 + ||\phi_m(t) - \phi_n(t)||^2\right) \leq$$
$$\leq |f_m(t) - f_n(t))| + |f_m(t) - f_n(t)|\, |\phi'_m(t) - \phi'_n(t)|^2.$$

Integrating, we obtain

$$|\phi'_m(t) - \phi'_n(t)|^2 + ||\phi_m(t) - \phi_n(t)||^2 \leq$$
$$\leq C\left(|\phi^1_m - \phi^1_n|^2 + ||\phi^0_m - \phi^0_n||^2 + \int_0^T |f_m(t) - f_n(t)|\, dt\right).$$

By convergence (2.7) it follows, from the above inequality, that

$$\max_{0\leq t\leq T} |\phi'_m(t) - \phi'_n(t)| \quad \text{converges to zero when} \quad m, n \to \infty$$
$$\max_{0\leq t\leq T} ||\phi_m(t) - \phi_n(t)|| \quad \text{converges to zero when} \quad m, n \to \infty$$

Then, $(\phi_\nu)_{\nu\in\mathbb{N}}$ is Cauchy sequence in $C^0([0,T]; H_0^1(\Omega))$ and $(\phi'_\nu)_{\nu\in\mathbb{N}}$ in $C^0([0,T]; L^2(\Omega))$. This implies that

$$\left| \begin{array}{l} \phi_\nu \quad \text{converges to } \ \xi \ \text{ in } \ C^0([0,T]; H_0^1(\Omega)) \\ \phi_\nu \quad \text{converges to } \ \zeta \ \text{ in } \ C^1([0,T]; L^2(\Omega)) \end{array} \right. \tag{2.32}$$

It follows by (2.10), that $\xi = \phi$, $\zeta = \phi'$, then we have the regularity (2.31).

Chapter 3

Hidden Regularity for Weak Solutions

3.1 Hidden regularity for weak solutions

In this section we study behavior of the normal derivative of the weak solution ϕ at the boundary Σ of the cylinder Q.

Consider a Hilbert space X with inner product $(\cdot,\cdot)$ and norm $|\cdot|$. If $v \in L^2(0,T;X)$ and the weak derivative $v' \in L^2(0,T;X)$, then $v \in C^0([0,T];X)$. It then follows that makes sense to define:

$$H_0^1(0,T;X) = \big\{v \in L^2(0,T;X), v' \in L^2(0,T;X);\ v(0) = v(T) = 0\big\}$$

with inner product

$$((u,v))_0 = \int_0^T (u(t),v(t))\,dt + \int_0^T (u'(t),v'(t))\,dt.$$

This is a Hilbert space.

By $\mathcal{D}(0,T;X)$ we represent the space of vector functions $\varphi\colon]0,T[\to X$, with compact support in $]0,T[$, infinitely derivable with the usual notion of convergence defined by Schwartz cf. Lions [32] or Medeiros-Miranda [48]. We represent by $H^{-1}(0,T;X)$ the dual of $H_0^1(0,T;X)$. We have the inclusions:

$$\mathcal{D}(0,T;X) \subset H_0^1(0,T;X) \subset L^2(0,T;X) \subset H^{-1}(0,T;X) \subset \mathcal{D}'(0,T;X).$$

By $\mathcal{D}'(0,T;X)$ we represent the dual of $\mathcal{D}(0,T;X)$ and we identify $L^2(0,T;X)$ to its dual. The above inclusions are continuous and each space is dense in the following. We prove that if $v \in L^2(0,T;X)$ then the weak derivative v' belongs to $H^{-1}(0,T;X)$ cf. Miranda [61].

When ϕ is an weak solution, cf. Chapter 2, then $\phi' \in L^2(0,T;L^2(\Omega))$. Then $\phi'' \in H^{-1}(0,T;L^2(\Omega))$. Therefore, $-\Delta\phi = f - \phi'' \in L^1(0,T;L^2(\Omega)) + H^{-1}(0,T;L^2(\Omega))$. When Γ is regular this implies that:

$$\phi \in L^1(0,T;H^2(\Omega)) + H^{-1}(0,T;H^2(\Omega))$$

and the normal derivative of ϕ has the regularity:

$$\frac{\partial\phi}{\partial\nu} \in L^1(0,T;H^{\frac{1}{2}}(\Gamma)) + H^{-1}(0,T;H^{\frac{1}{2}}(\Gamma)). \tag{3.1}$$

From (3.1) does not follows that $\dfrac{\partial\phi}{\partial\nu}$ belongs to $L^2(0,T;L^2(\Gamma))$ and is bounded in this space. We shall prove, by method of multiplies, that, in fact, $\dfrac{\partial\phi}{\partial\nu}$ is bounded in the norm of $L^2(0,T;L^2(\Gamma))$. By the reason that this regularity for $\dfrac{\partial\phi}{\partial\nu}$ does not comes from the properties of the weak solution ϕ given by Theorem 2.1 of Chapter 2, is that Lions [37] called it **Hidden Regularity** of $\dfrac{\partial\phi}{\partial\nu}\cdot$ This regularity was proved by Lions, first time, in 1983 in the reference [34].

Lemma 3.1 *Let be $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ the vector field of exterior normals to Γ. Then there exists a vector field $h = (h_1, h_2, \ldots, h_n) \in [C^1(\overline{\Omega})]^n$ such that*

$$h_i = \nu_i \ \ \textit{on } \Gamma \ \textit{for} \ \ i = 1, 2, \ldots, n.$$

Proof: We know by Sobolev's embedding theorem that for $m > 1 + \dfrac{n}{2}$ we have $H^m(\Omega) \subset C^1(\overline{\Omega})$ continuously. The trace operator γ_0 is a bijection between $H^m(\Omega)$ and $H^{m-\frac{1}{2}}(\Gamma)$. Therefore, if $\nu_k \in H^{m-\frac{1}{2}}(\Gamma)$ there exists $h_k \in H^m(\Omega) \subset C^1(\overline{\Omega})$ such that $\gamma_0 h_k = \nu_k$. ■

Lemma 3.2 *If $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$, then*

$$\frac{\partial\phi}{\partial x_i} = \nu_i \frac{\partial\phi}{\partial\nu} \quad \textit{on} \quad \Gamma \tag{3.2}$$

$$|\nabla\phi|^2 = \left(\frac{\partial\phi}{\partial\nu}\right)^2. \tag{3.3}$$

Proof: Let us prove (3.2), that is, we prove that:

$$\int_\Gamma \frac{\partial\phi}{\partial x_i}\,\theta d\Gamma = \int_\Gamma \nu_i \frac{\partial\phi}{\partial\nu}\,\theta d\Gamma \quad \text{for all} \quad \theta \in \mathcal{D}(\Gamma).$$

In fact, let be $\xi \in C^2(\overline{\Omega})$ such that $\gamma_0 \xi = \theta$. Such ξ exists by the embedding $H^m(\Omega) \subset C^2(\overline{\Omega})$ for $m > 2 + \frac{2}{n}$ and the trace theorem. Let be $(h_k)_{1 \leq k \leq n}$ the vector field of Lemma 3.1. From Gauss'formula, we obtain:

$$\int_\Omega \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} (\phi h_j \xi)\, dx = \int_\Gamma \nu_i \frac{\partial}{\partial x_j} (\phi h_j \xi)\, d\Gamma. \tag{3.4}$$

The integral on the right hand side of (3.4) is

$$\int_\Gamma \nu_i \frac{\partial \phi}{\partial x_j} h_j \xi \, d\Gamma$$

because $\phi \in H_0^1(\Omega) \cap H^2(\Omega)$. Whence,

$$\int_\Omega \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\phi h_j \xi)\, dx = \int_\Gamma \nu_i \frac{\partial \phi}{\partial x_j} h_j \xi \, d\Gamma = \int_\Gamma \nu_i \frac{\partial \phi}{\partial x_j} \nu_j \theta \, d\Gamma,$$

because $h_j = \nu_j$ and $\xi = \theta$ on Γ, by definition. Adding the above equality from $j = 1$ up to $j = n$, we obtain:

$$\sum_{i=1}^{n} \int_\Omega \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\phi h_j \xi)\, dx = \sum_{i=1}^{n} \int_\Gamma \nu_i \frac{\partial \phi}{\partial \nu} v_j \theta \, d\Gamma.$$

By application of Gauss lemma to the left hand side, we obtain:

$$\int_\Omega \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} (\phi h_j \xi)\, dx = \int_\Gamma \frac{\partial \phi}{\partial x_i} \theta \nu_j^2 \, d\Gamma.$$

Adding from $j = 1$ to $j = n$, we obtain:

$$\int_\Gamma \frac{\partial \phi}{\partial x_i} \theta \, d\Gamma = \int_\Gamma \nu_i \frac{\partial \phi}{\partial \nu} \theta \, d\Gamma$$

for all $\theta \in \mathcal{D}(\Gamma)$. ■

To prove (3.3) it is sufficient to consider

$$\nu_i \frac{\partial \phi}{\partial x_i} \nu_i \frac{\partial \phi}{\partial x_i} = \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_i} = |\nabla \phi|^2.$$

■

Note that repeated index means summation.

Lemma 3.3 *Let $(q_k)_{1 \leq k \leq n}$ be a vector field such that $q_k \in C^1(\overline{\Omega})$ for $1 \leq k \leq n$. If $(\phi_n)_{n \in \mathbb{N}}$ is a sequence of strong solutions of (*), cf. Chapter 1, then, for each $n \in \mathbb{N}$ it is true the identity:*

$$\begin{aligned} \frac{1}{2} \int_\Sigma h_k \nu_k \left(\frac{\partial \phi_n}{\partial \nu} \right)^2 d\Gamma\, dt = & \left(\phi_n'(t), q_k \frac{\partial \phi_n(t)}{\partial x_k} \right) \Big|_0^T + \\ & + \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \left[|\phi_n'(t)|^2 - |\nabla \phi_n(t)|^2 \right] dxdt + \\ & + \int_Q \frac{\partial q_k}{\partial x_j} \frac{\partial \phi_n}{\partial x_k} \frac{\partial \phi_n}{\partial x_j} dxdt - \int_Q f_n q_k \frac{\partial \phi_n}{\partial x_k} dxdt \end{aligned} \tag{3.5}$$

where $q_k \in C^1(\overline{\Omega})$, for $1 \leq k \leq n$.

Proof: We use the notation

$$X = \left(\phi_n'(t), q_k \frac{\partial \phi_n(t)}{\partial x_k}\right)\Bigg|_0^T = \left(\phi_n'(T), q_k \frac{\partial \phi_n(T)}{\partial x_k}\right) - \left(\phi_n'(0), q_k \frac{\partial \phi_n(0)}{\partial x_k}\right).$$

Note that $q_k \dfrac{\partial \phi_n}{\partial x_k} \in L^2(Q)$ because ϕ_n is strong solution, see Chapter 1.

Then it makes sense multiply both sides of $\phi_n'' - \Delta\phi_n = f_n$, a.e. in Q, by $q_k \dfrac{\partial \phi_n}{\partial x_k}$ and integrate on Q. We have:

$$\int_Q \phi_n'' q_k \frac{\partial \phi_n}{\partial x_k}\, dxdt - \int_Q \Delta\phi_n q_k \frac{\partial \phi_n}{\partial x_k}\, dxdt = \int_Q f_n q_k \frac{\partial \phi_n}{\partial x_k}\, dxdt \tag{3.6}$$

where double index means addition on $1 \le k \le n$.

Analysis of $\displaystyle\int_Q \Delta\phi_n q_k \frac{\partial \phi}{\partial x_k}\, dxdt.$

To make easy the notation we use ϕ instead of ϕ_n . We obtain:

$$-\int_Q \Delta\phi q_k \frac{\partial \phi}{\partial x_k}\, dxdt = -\int_0^T \int_\Omega \Delta\phi q_k \frac{\partial \phi}{\partial x_k}\, dxdt. \tag{3.7}$$

By Gauss'formula we obtain:

$$-\int_\Omega \Delta\phi q_k \frac{\partial \phi}{\partial x_k}\, dx = -\int_\Gamma \frac{\partial \phi}{\partial \nu}\, q_k \frac{\partial \phi}{\partial x_k}\, d\Gamma + \int_\Omega \nabla\phi \cdot \nabla\left(q_k \frac{\partial \phi}{\partial x_k}\right) dx. \tag{3.8}$$

We have:

$$\begin{aligned} \nabla\phi \cdot \nabla\left(q_k \frac{\partial \phi}{\partial x_k}\right) &= \frac{\partial \phi}{\partial x_i}\, q_k \frac{\partial}{\partial x_k}\left(\frac{\partial \phi}{\partial x_i}\right) + \\ + \frac{\partial \phi}{\partial x_i} \frac{\partial q_k}{\partial x_i} \frac{\partial \phi}{\partial x_k} &= \frac{1}{2}\, q_k \frac{\partial}{\partial x_k}\left(\frac{\partial \phi}{\partial x_i}\right)^2 + \\ + \frac{\partial \phi}{\partial x_i} \frac{\partial q_k}{\partial x_i} \frac{\partial \phi}{\partial x_k} &= \frac{1}{2}\, q_k \frac{\partial}{\partial x_k}\, |\nabla\phi|^2 + \frac{\partial \phi}{\partial x_i} \frac{\partial q_k}{\partial x_i} \frac{\partial \phi}{\partial x_k}\,. \end{aligned} \tag{3.9}$$

By (3.9) we modify the last integral in the right hand side of (3.8) obtainning:

$$\begin{aligned} -\int_\Omega \Delta\phi q_k \frac{\partial \phi}{\partial x_k}\, dx &= -\int_\Gamma \frac{\partial \phi}{\partial \nu}\, q_k \frac{\partial \phi}{\partial x_k}\, d\Gamma + \\ + \frac{1}{2}\int_\Omega q_k \frac{\partial}{\partial x_k}\, |\nabla\phi|^2\, dx &+ \int_\Omega \frac{\partial q_k}{\partial x_i} \frac{\partial \phi}{\partial x_k} \frac{\partial \phi}{\partial x_i}\, dx. \end{aligned} \tag{3.10}$$

By Gauss lemma, we obtain:

$$\frac{1}{2}\int_\Omega q_k \frac{\partial}{\partial x_k}\, |\nabla\phi|^2\, dx = \frac{1}{2}\int_\Gamma q_k |\nabla\phi|^2\, \nu_k\, d\Gamma - \frac{1}{2}\int_\Omega \frac{\partial q_k}{\partial x_k}\, |\nabla\phi|^2\, dx. \tag{3.11}$$

Note by Lemma 3.2, $|\nabla\phi|^2 = \left(\dfrac{\partial\phi}{\partial\nu}\right)^2$ and $\dfrac{\partial\phi}{\partial x_k} = \nu_k \dfrac{\partial\phi}{\partial\nu}\cdot$ Consequently, substituting (3.11) in (3.10) and integrating on $]0,T[$, we obtain:

$$\begin{aligned} &-\int_Q \Delta\phi q_k \frac{\partial\phi}{\partial x_k}\,dx = -\frac{1}{2}\int_\Sigma q_k\nu_k\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt - \\ &-\frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\,|\nabla\phi|^2\,dxdt + \int_Q \frac{\partial q_k}{\partial x_i}\,\frac{\partial\phi}{\partial x_k}\,\frac{\partial\phi}{\partial x_i}\,dxdt. \end{aligned} \tag{3.12}$$

Analysis of $\displaystyle\int_Q \phi'' q_k \frac{\partial\phi}{\partial x_k}\,dxdt.$

We have:

$$\int_Q \phi'' q_k \frac{\partial\phi}{\partial x_k}\,dxdt = \int_0^T\int_\Omega \phi'' q_k \frac{\partial\phi}{\partial x_k}\,dxdt.$$

We have:

$$\int_0^T \frac{\partial}{\partial t}\left(\phi'\cdot q_k\frac{\partial\phi}{\partial x_k}\right)dt = \int_Q \phi'' q_k\frac{\partial\phi}{\partial x_k}\,dxdt + \int_Q \phi' q_k \frac{\partial\phi'}{\partial w_k}\,dxdt.$$

Whence,

$$\int_Q \phi'' q_k \frac{\partial\phi}{\partial x_k}\,dxdt = \left(\phi'(t)\cdot q_k\frac{\partial\phi(t)}{\partial x_k}\right)\Bigg|_0^T - \frac{1}{2}\int_Q q_k\frac{\partial}{\partial x_k}\,\phi'^2\,dxdt. \tag{3.13}$$

Note that $\phi' \in C^0([0,T];H_0^1(\Omega))$, by regularity of strong solutions, cf. Chapter 1. Then,

$$\int_\Omega \frac{\partial}{\partial x_k}\left(\frac{q_k}{2}\,\phi'^2\right)dx = \int_\Gamma \frac{q_k}{2}\,\phi'^2\nu_k\,d\Gamma = 0.$$

It follows from the above equality that:

$$-\frac{1}{2}\int_Q q_k\frac{\partial}{\partial x_k}\,\phi'^2\,dxdt = \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\,\phi'^2\,dxdt. \tag{3.14}$$

From (3.14) we obtain from (3.13):

$$\int_Q \phi'' q_k \frac{\partial\phi}{\partial x_k}\,dxdt = \left(\phi'(t), q_k\frac{\partial\phi(t)}{\partial x_k}\right)\Bigg|_0^T + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\,\phi'^2\,dxdt. \tag{3.15}$$

Substituting (3.12) and (3.15) in (3.6) we obtain (3.5), after substituting ϕ_n instead of ϕ.

■

We define the energy associated to ϕ_n by

$$E_n(t) = \frac{1}{2}\int_\Omega \left(\phi_n'^2(t) + |\nabla\phi_n(t)|^2\right)dx.$$

If $t=0$, we have:

$$E_n(0) = \frac{1}{2}\int_\Omega \left(|\phi_n^1|^2 + |\nabla\phi_n^0|^2\right) dx.$$

From the energy inequality (3.17), Chapter 1, we have:

$$E_n(t) \leq C\left(E_n(0) + \int_0^T |f_(s)|\, ds\right). \tag{3.16}$$

Evidently a similar inequality is true for weak solutions.

As a consequence of the identity (3.5) of Lemma 3.3, we obtain a key estimate for $\dfrac{\partial\phi_n}{\partial\nu}$ on Σ. In fact, take $q_k = h_k$ the vector field of Lemma 3.1. Then, from (3.5) we obtain:

$$\begin{aligned}\frac{1}{2}\int_\Sigma \left(\frac{\partial\phi_n}{\partial\nu}\right)^2 d\Gamma dt &= \frac{1}{2}\int_Q \frac{\partial h_k}{\partial x_k}\left(|\phi_n'|^2 - |\nabla\phi_n|^2\right) dxdt + \\ &+ X + \int_Q \frac{\partial h_k}{\partial x_i}\frac{\partial\phi_n}{\partial x_k}\frac{\partial\phi_n}{\partial x_i}\, dxdt - \int_Q f_n h_k \frac{\partial\phi_n}{\partial x_k}\, dxdt.\end{aligned} \tag{3.17}$$

Since $h_k \in C^1(\overline{\Omega})$ and $f_n \in C^0([0,T]; C^1(\overline{\Omega}))$, look Chapter 2, we have:

$$\frac{1}{2}\left|\int_Q \frac{\partial h_k}{\partial x_k}\left(|\phi_n'|^2 - |\nabla\phi_n|^2\right) dxdt\right| \leq C\, E_n(t),$$

$$\left|\int_Q f_n h_k \frac{\partial\phi_n}{\partial x_k}\, dxdt\right| \leq C\sum_{k=1}^n \int_\Omega \left(\frac{\partial\phi_n}{\partial x_k}\right)^2 dx \leq C\, E_n(t),$$

where C represents different constants

$$\left|\int_Q \frac{\partial h_k}{\partial x_i}\frac{\partial\phi_n}{\partial x_k}\frac{\partial\phi_n}{\partial x_i}\, dxdt\right| \leq C\int_\Omega |\nabla\phi_n|^2\, dxdt \leq C\, E_n(t).$$

By Schwarz and inequality $2ab \leq a^2 + b^2$ we obtain:

$$|X| \leq 2\sup_{0\leq t\leq T}\left|\left(\phi_n'(t), h_k\frac{\partial\phi_n(t)}{\partial x_k}\right)\right| \leq C\sup_{0\leq t\leq T} E_n(t).$$

Then, by (3.16) the inequality (3.17) becomes:

$$\frac{1}{2}\int_\Sigma \left(\frac{\partial\phi_n}{\partial\nu}\right)^2 d\Gamma dt \leq C_1\left(E(0) + \int_0^T |f_n(s)|\, ds\right) \tag{3.18}$$

where

$$E_0 = E(0) = \frac{1}{2}\int_\Omega \left(|\phi^1|^2 + |\nabla\phi^0|^2\right) dx. \tag{3.19}$$

From (3.18) it follows that the sequence $\left(\dfrac{\partial\phi_n}{\partial\nu}\right)_{n\in\mathbb{N}}$ is bounded in $L^2(\Sigma)$. Then there exists a subsequence, still represented with the same index, such that:

$$\frac{\partial\phi_n}{\partial\nu} \quad \text{converges to}\, \chi \;\text{ weakly on }\; L^2(\Sigma) \tag{3.20}$$

and

$$|\chi|_{L^2(\Sigma)} \leq \varliminf_{n\to\infty} \left|\frac{\partial \phi_n}{\partial \nu}\right|_{L^2(\Sigma)}.$$

We choose $(\phi_n)_{n\in\mathbb{N}}$ as the sequence of strong solutions which approximate the weak solution ϕ as done in Chapter 2. Then we can formulate the hidden regularity by the following.

Theorem 3.1 (Hidden Regularity) *If ϕ is the weak solution of (*), Chapter 1, then we have:*

$$\frac{\partial \phi}{\partial \nu} \in L^2(\Sigma) \tag{3.21}$$

$$\int_\Sigma \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma dt \leq C\left(E_0 + \int_0^T |f(s)|\, ds\right). \tag{3.22}$$

Proof: To prove this theorem, it is sufficient to show that the limit χ obtained in (3.20) is equal to $\dfrac{\partial \phi}{\partial \nu}\cdot$ Then (3.18) implies (3.21).

In fact, let us prove that $\chi = \dfrac{\partial \phi}{\partial \nu}\cdot$ We know the weak solution ϕ is the weak limit of the approximated strong solution $(\phi_n)_{n\in\mathbb{N}}$, cf. Chapter 2.

Note that γ_1 is the trace of the normal derivative. We need to prove that $\gamma_1\phi_n \rightharpoonup \gamma_1\phi$ in a topology that implies the convergence in $L^2(\Sigma)$.

We begin observing that:

$$-\Delta\phi_n = f_n = \phi_n'' \quad \text{in} \quad \mathcal{D}'(0,T;H^{-1}(\Omega)). \tag{3.23}$$

We have $f_n \in C^0([0,T];C^1(\overline{\Omega}))$ and $\phi_n' \in L^2(0,T;H_0^1(\Omega))$. Then, exists z_n, w_n in $L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))$ such that:

$$\left|\begin{array}{lll} -\Delta w_n = f_n & \text{and} & ||w_n||_{L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))} \leq C|f_n|_{L^2(Q)}, \\ -\Delta z_n = \phi_n' & \text{and} & ||z_n||_{L^2(0,T;H_0^1(\Omega)\cap H^2(\Omega))} \leq C|\phi_n'|_{L^2(Q)} \end{array}\right. \tag{3.24}$$

by results of elliptic equation. By (3.24) we change (3.23) obtaining:

$$-\Delta\phi_n = -\Delta w_n - (\Delta z_n)' \quad \text{in} \quad \mathcal{D}'(0,T;H^{-1}(\Omega)). \tag{3.25}$$

We will prove that (3.25) implies:

$$\phi_n = -w_n - z_n' \quad \text{in} \quad \mathcal{D}'(0,T;H^{-1}(\Omega)).$$

In fact, by (3.25), for each $\theta \in \mathcal{D}(0,T)$ we have:

$$-\int_0^T \Delta\phi_m\theta\, dx = -\int_0^T \Delta w_n\theta\, dx + \int_0^T \Delta z_n\theta'\, dt \quad \text{in} \quad H^{-1}(\Omega).$$

We know that $\Delta \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$, we obtain:

$$-\Delta\left(\int_0^T \phi_n \theta \, dt\right) = \Delta\left[-\int_0^T w_n \theta \, dt + \int_0^T z_n \theta' \, dt\right].$$

By the uniqueness of Dirichlet problem we obtain:

$$\int_0^T \phi_n \theta \, dt = -\int_0^T w_n \theta \, dt + \int_0^T z_n \theta' \, dt \quad \text{in} \quad H^{-1}(\Omega),$$

for all $\theta \in \mathcal{D}(0,T)$, that is,

$$\phi_n = -w_n - z_n' \quad \text{in} \quad \mathcal{D}'(0,T;H^{-1}(\Omega)). \tag{3.26}$$

But, since $z_n \in L^2(0,T;H^2(\Omega))$ it implies that $z_n' \in H^{-1}(0,T;H^2(\Omega))$ and $\gamma_1 z_n' \in H^{-1}(0,T;H^{1/2}(\Gamma))$.

$$\gamma_1 \phi_n = -\gamma_1 w_n - \gamma_1 z_n' \quad \text{in} \quad H^{-1}(0,T;H^{1/2}(\Gamma)). \tag{3.27}$$

By $(3.24)_1$, since $f_n \to f$ in $L^1(0,T;H^1(\Omega))$ or f_n is bounded in $L^2(Q)$, it implies $||w_n||_{L^2(0,T;H^2(\Omega))}$ is bounded. Consequently, there exists a sequence $(w_n)_{n\in\mathbb{N}}$, such that

$$w_n \rightharpoonup \psi \quad \text{weak in} \quad L^2(0,T;H^2(\Omega)).$$

Note that $\Delta w_n = -f_n$. But $f_n \to f$ in $L^1(0,T;H_0^1(\Omega))$ and if w is such that $\Delta w = -f$, we obtain $\psi = w$. From the continuity of the trace γ_1 we obtain:

$$\gamma_1 w_n \rightharpoonup \gamma_1 w \quad \text{weak in} \quad L^2(0,T;H^{1/2}(\Gamma)). \tag{3.28}$$

By $(3.24)_2$, since ϕ_n' is bounded in $L^2(Q)$, we obtain, by similar argument, a subsequence $(z_n)_{n\in\mathbb{N}}$ such that

$$\gamma_1 z_n \rightharpoonup \gamma_1 z \quad \text{in} \quad L^2(0,T;H^{1/2}(\Gamma)). \tag{3.29}$$

We prove that

$$\gamma_1 z_n' \rightharpoonup \gamma_1 z' \quad \text{weak in} \quad H^{-1}(0,T;H^{1/2}(\Gamma)). \tag{3.30}$$

Note that by (3.24) w and z are solutions of $\Delta w = -f$ and $\Delta z = -\phi'$ with ϕ' weak limit of ϕ_n' where ϕ is the weak solution. By $-\Delta\phi = f - \phi''$ in $\mathcal{D}'(0,T;H^{-1}(\Omega))$, we obtain $\phi = -w - z'$ and $\gamma_1\phi = -\gamma_1 w - \gamma_1 z'$ in $H^{-1}(0,T;H^{1/2}(\Gamma))$. We have by (3.28) and (3.29)

$$\gamma_1\phi_n = -\gamma_1 w_n - \gamma_1 z_n' \to -\gamma_1 w - \gamma_1 z' = \gamma_1 \phi \quad \text{in} \quad H^{-1}(0,T;H^{1/2}(\Gamma))$$

or

$$\langle \gamma_1 \phi_n, v\rangle \to \langle \gamma_1\phi, v\rangle \quad \text{for all} \quad v \in H_0^1(0,T;H^{1/2}(\Gamma)).$$

We obtained by (3.20):

$$\langle \gamma_1 \phi_n, v\rangle \to \langle \chi, v\rangle \quad \text{for all} \quad v \in L^2(0,T;L^2(\Gamma)).$$

Since $H_0^1(0,T;H^{1/2}(\Gamma)) \subset L^2(0,T;L^2(\Gamma))$ we have $\chi = \gamma_1\phi$. ■

Chapter 4

Ultra Weak Solutions

4.1 Ultra Weak Solutions

In this section we are interested in the study of the non homogeneous boundary value problem:

$$\left| \begin{array}{l} z'' - \Delta z = 0 \quad \text{in} \quad Q, \\ z = v \quad \text{on} \quad \Sigma, \\ z(0) = z^0, \ z'(0) = z^1 \quad \text{in} \quad \Omega \end{array} \right. \tag{4.1}$$

when z^0, z^1 are not regular as in Chapter 1 and 2. This type of problem was analysed, first time, in Lions-Magenes [43], cf. also Lions [39]. One of the questions is an appropriate definition of what we understand by solution of (4.1). As the initial values z^0, z^1 are not regular, we define the solution of (4.1) by the so called **Transposition Method**, as proposed in Lions-Magenes op. cit. Here we follows an heuristic method in order to find a natural definition of what we will call **ultra weak solution** as defined by Lions-Magenes. In fact, multiply both sides of the equation $(4.1)_1$ by a function $\theta = \theta(x,t)$, $x \in \Omega$, $t \in]0,T[$ and integrate in Q, formally, by parts in t.

$$\begin{aligned} &\int_0^T\int_\Omega z(\theta'' - \Delta\theta)\,dxdt + \int_\Omega z'(x,T)\theta(x,T)\,dx - \\ &- \int_\Omega z'(x,0)\theta'(x,0)\,dx - \int_\Omega z(x,T)\theta'(x,T)\,dx + \int_\Omega z(x,0)\theta'(x,0)\,dx - \\ &- \int_0^T\int_\Gamma \frac{\partial z}{\partial\nu}\,\theta\,d\Gamma dt + \int_0^T\int_\Gamma \frac{\partial \theta}{\partial\nu}\,z\,d\Gamma dt = 0. \end{aligned} \tag{4.2}$$

We have no information, up to know, about $z(x,T)$, $z'(x,T)$ and $\dfrac{\partial z}{\partial \nu}\cdot$ Then, we choose $\theta = \theta(x,t)$ such that

$$\theta(x,T) = 0, \ \theta'(x,T) = 0 \quad \text{and} \quad \theta(x,t) = 0 \quad \text{on} \quad \Sigma. \tag{4.3}$$

Whence for this choice of $\theta = \theta(x,t)$, the equality (4.2) turns out:

$$\langle z, \theta'' - \Delta\theta \rangle = -\langle z^0, \theta'(0)\rangle + \langle z^1, \theta(0)\rangle - \left\langle \frac{\partial \theta}{\partial \nu}, v \right\rangle \tag{4.4}$$

where $\langle \cdot, \cdot \rangle$ represents different pairs of duality.

The definition of ultra weak solution, by transposition method, will be given as a functional defined by the expression (4.4). Then we will see that is natural to choose $\theta = \theta(x,t)$ as the weak solution of the backward problem:

$$\left| \begin{array}{l} \theta'' - \Delta\theta = f \quad \text{in} \quad Q, \\ \theta = 0 \quad \text{on} \quad \Sigma, \\ \theta(T) = 0, \ \theta'(T) = 0 \quad \text{in} \quad \Omega. \end{array} \right. \tag{4.5}$$

If we take $f \in L^1(0,T; L^2(\Omega))$ and consider the change of variables $T - t$ instead of t in (4.5), then (4.5) is a particular case of the problem studied in Chapter 2 for weak solutions. Then we can apply to θ, weak solution of (4.5), all the conclusions of Chapters 2 and 3. Then, we have:

$$\left| \begin{array}{l} \theta \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)), \\ \dfrac{\partial \theta}{\partial \nu} \in L^2(\Sigma). \end{array} \right. \tag{4.6}$$

By Chapter 2, Corollary 2.1, inequality (2.30), since $\theta^0 = \theta^1 = 0$, we obtain:

$$||\theta'||_{L^\infty(0,T;L^2(\Omega))} + ||\theta||_{L^\infty(0,T;H_0^1(\Omega))} \leq C||f||_{L^1(0,T;L^2(\Omega))} \,. \tag{4.7}$$

By Chapter 3, Theorem 3.1, we obtain:

$$\left| \begin{array}{l} \dfrac{\partial \theta}{\partial \nu} \in L^2(\Sigma), \\ \left\| \dfrac{\partial \theta}{\partial \nu} \right\|_{L^2(\Sigma)} \leq C||f||_{L^1(0,T;L^2(\Omega))}. \end{array} \right. \tag{4.8}$$

As a consequence of (4.6) we have $\theta' \in L^2(\Omega)$, $\theta(0) \in H_0^1(\Omega)$ and $\dfrac{\partial \theta}{\partial \nu} \in L^2(\Sigma)$. Then, in order to ensure that the right hand side of (4.4) makes sense it is sufficient to choose:

$$z^0 \in L^2(\Omega), \ z^1 \in H^{-1}(\Omega) \quad \text{and} \quad v \in L^2(\Sigma). \tag{4.9}$$

Motivated by the expression (4.6) and by the above considerations, for each set $\{z^0, z^1, v\}$ in the class (4.9) is well defined the functional S on $L^1(0,T; L^2(\Omega))$ by:

$$\langle S, f \rangle = -(z^0, \theta'(0)) + \langle z^1, \theta(0)\rangle - \int_\Sigma \frac{\partial \theta}{\partial \nu} v \, d\Gamma dt \tag{4.10}$$

for all solution θ of the problem (4.5).

From the estimates (4.7) and $(4.8)_2$, for the weak solution θ of (4.5), we obtain, from (4.10):

$$\begin{aligned} |\langle S, f\rangle| &\leq |z^0|\,|\theta'(0)| + ||z^1||_{H^{-1}(\Omega)}\,||\theta(0)|| + \left\|\tfrac{\partial\theta}{\partial\nu}\right\|_{L^2(\Sigma)}\,||v||_{L^2(\Sigma)} \leq \\ &\leq C\big(|z^0| + ||z^1||_{H^{-1}(\Omega)} + ||v||_{L^2(\Sigma)}\big)||f||_{L^1(0,T;L^2(\Omega))}\,. \end{aligned} \tag{4.11}$$

Therefore, $S\colon L^1(0,T;L^2(\Omega)) \to \mathbb{R}$, defined by (4.10), is a linear form which is continuous by (4.11). It follows that S is an object of $L^\infty(0,T;L^2(\Omega))$, the topological dual of $L^1(0,T;L^2(\Omega))$. Furthermore,

$$||S||_{L^\infty(0,T;L^2(\Omega))} \leq C\big(|z^0| + ||z^1||_{H^{-1}(\Omega)} + ||v||_{L^2(\Sigma)}\big). \tag{4.12}$$

Definition 4.1 *For $\{z^0, z^1, v\} \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma)$, we call ultra weak solution of the non homogeneous mixed problem (4.1), a function $z \in L^\infty(0,T;L^2(\Omega))$ satisfying the condition:*

$$\int_Q zf\,dxdt = -(z^0, \theta(0)) + \langle z^1, \theta(0)\rangle - \int_\Sigma \frac{\partial\theta}{\partial\nu}\,v\,d\Gamma dt, \tag{4.13}$$

for all $f \in L^1(0,T;L^2(\Omega))$, with θ solution of the backward problem (4.5).

We say that the ultra weak solution z of (4.1) is defined by transposition. For this reason, we sometimes call it **solution by transposition** instead of **ultra weak solution**.

Theorem 4.1 (Existence and Uniqueness) *Exists only one ultra weak solution z of the non homogeneous mixed problem (4.1). Furthermore, z satisfies:*

$$||z||_{L^\infty(0,T;L^2(\Omega))} \leq C\big(|z^0| + ||z^1||_{H^{-1}(\Omega)} + ||v||_{L^2(\Sigma)}\big). \tag{4.14}$$

Note that the constant C in (4.14) depends only of $T > 0$ and the vector field $(h_k)_{k\in\mathbb{N}}$ introduced in Chapter 3.

Proof: The existence is a consequence of (4.10), (4.11) and Riesz representation theorem for the objects of $L^\infty(0,T;L^2(\Omega))$ dual of $L^1(0,T;L^2(\Omega))$. The uniqueness is a consequence of Du Bois Raymond's Lemma (cf. Medeiros-Miranda [48]). ■

Corollary 4.1 *The linear function $\{z^0, z^1, v\} \to z$, from $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma)$ into $L^\infty(0,T;L^2(\Omega))$ is continuous, where z is the ultra weak solution of (4.1) with data z^0, z^1, v.*

In the applications it is important to know the regularity of the ultra weak solutions as we have seen in the strong and weak cases.

Theorem 4.2 (Regularity of Ultra Weak Solutions) *The ultra weak solution z of (4.1) belongs to the class:*

$$z \in C^0([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)) \tag{4.15}$$

and satisfies the estimate:

$$||z||_{L^\infty(0,T;L^2(\Omega))} + ||z'||_{L^\infty(0,T;H^{-1}(\Omega))} \leq C(|z^0| + ||z^1||_{H^{-1}(\Omega)} + ||v||_{L^2(\Sigma)}), \tag{4.16}$$

where $C > 0$ is a constant which depends only of T and the vector field $(h_k)_{k\in\mathbb{N}}$.

Proof: We divide the proof in two parts. In the first we prove regularity for z and then for z'.

Step 1. Let us recall, first of all, some results of regularity for strong solutions. As we have proved in Chapter 1, Corollary 1.1, from inequality (1.25), if ϕ is an strong solutions of (*) of Chapter 1, then

$$\phi \in C^0([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)) \tag{4.17}$$

and

$$\begin{aligned} &||\phi||_{L^\infty(0,T;H_0^1(\Omega)\cap H^2(\Omega))} + ||\phi'||_{L^\infty(0,T;H_0^1(\Omega))} \leq \\ &\leq 2(||\phi^1|| + ||\phi^0||_{H_0^1(\Omega)\cap H^2(\Omega)} + ||f||_{L^1(0,T;H_0^1(\Omega))}). \end{aligned} \tag{4.18}$$

Note that $H_0^1(\Omega) \cap H^2(\Omega)$ is equipped with the norm of the Laplace operator.

Let us consider the system (4.1) in the regular case, that is when:

$$z^0 \in H_0^1(\Omega),\ z^1 \in L^2(\Omega) \quad \text{and} \quad v \in H_0^2(0,T; H^{3/2}(\Gamma)). \tag{4.19}$$

Lemma 4.1 *Exists only one weak solution z of the non homogeneous mixed problem (4.1) when we choose the initial data (4.19) and z has the regularity:*

$$z \in C^0([0,T]; H_0^1(\Omega)) \cap C^1([0,T; L^2(\Omega)). \tag{4.20}$$

Furthermore, this weak solution z is an ultra weak solution.

Proof: In fact, let $\widehat{v} \in H_0^2(0,T;H^2(\Omega))$ such that $\widehat{v} = 0$ on Σ. Note that $\widehat{v}''$ and $\Delta\widehat{v}$ are objects of $L^2(0,T;L^2(\Omega))$. Let us consider the mixed problem:

$$\left|\begin{array}{l} u'' - \Delta u = -\widehat{v}'' + \Delta\widehat{v} \quad \text{in} \quad Q, \\ u = 0 \quad \text{on} \quad \Sigma, \\ u(0) = z^0 \quad \text{and} \quad u'(0) = z^1 \quad \text{in} \quad \Omega. \end{array}\right. \tag{4.21}$$

Since $-\widehat{v}'' + \Delta\widehat{v}$ is in $L^2(0,T;L^2(\Omega))$, $z^0 \in H_0^1(\Omega)$ and $z' \in L^2(\Omega)$, it follows, by regularity of weak solutions, cf. Chapter 2, Theorem 3.1, that the solution u of (4.21) has the regularity:

$$u \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)).$$

By definition of weak solution, u satisfies:

$$\frac{d}{dt}\,(u'(t),\psi) + ((u(t),\psi)) = (-\widehat{v}'' + \Delta\widehat{v},\psi)$$

for all $\psi \in H_0^1(\Omega)$ in the sense of $\mathcal{D}'(0,T)$. Then

$$\frac{d}{dt}\,(u'(t) + \widehat{v}'(t),\psi) + ((u(t) + \widehat{v}(t),\psi)) = 0$$

for all $\psi \in H_0^1(\Omega)$ in the sense of $\mathcal{D}'(0,T)$.

We have $u+\widehat{v} = v$ on Σ and $(u+\widehat{v})(0) = z^0$ and $(u+\widehat{v})'(0) = z^1$. Therefore, $z = u+\widehat{v}$ is an weak solution of the problem (4.1) with initial data (4.19). Consequently by regularity of weak solution we have $z \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$ and we have uniqueness too.

To complete the proof we need to prove that z is also ultra weak solution of (4.1). In fact, let be $f \in L^1(0,T;L^2(\Omega))$ and consider the sequence $(f_\mu)_{\mu\in\mathbb{N}}$, with $f_\mu \in L^1(0,T;H_0^1(\Omega))$ such that

$$\lim_{\mu\to\infty} f_\mu = f \quad \text{in} \quad L^1(0,T;L^2(\Omega)). \tag{4.22}$$

Let us consider the two backwards problems:

$$\left|\begin{array}{l} \theta_\mu'' - \Delta\theta_\mu = f_\mu \quad \text{in} \quad Q, \\ \theta_\mu = 0 \quad \text{on} \quad \Sigma, \\ \theta_\mu(T) = 0,\ \theta_\mu'(T) = 0 \quad \text{on} \quad \Omega \end{array}\right. \tag{4.23}$$

and

$$\left|\begin{array}{l} \theta'' - \Delta\theta = f \quad \text{in} \quad Q, \\ \theta = 0 \quad \text{on} \quad \Sigma, \\ \theta(T) = 0,\ \theta'(T) = 0 \quad \text{in} \quad \Omega. \end{array}\right. \tag{4.24}$$

By the regularity of f_μ and f, it follows that exists strong solution θ_μ of (4.23) and weak solution θ of (4.24) and:

$$\theta_\mu \in C^0([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)). \tag{4.25}$$

It follows that $\theta_\mu - \theta$ is weak solution of a backward homogeneous problem of the type (4.24). Then changing t in $T-t$, we have, by energy inequality Chapter 2, Theorem 2.2, (2.29) and Chapter 3, Theorem 3.1, hidden regularity (4.21):

$$|\theta'_\mu(T-t) - \theta'(T-t)|^2 + ||\theta'_\mu(T-t) - \theta(T-t)||^2 + \left\| \frac{\partial \theta_\mu}{\partial \nu} - \frac{\partial \theta}{\partial \nu} \right\|^2_{L^2(\Sigma)} \leq$$
$$\leq C||f_\mu - f||_{L^1(0,T;L^2(\Omega))}\,,$$

for all $0 \leq t \leq T$. Taking $t = T$ and let be $\mu \to \infty$, we obtain, from the last inequality:

$$\left|\begin{array}{l} \theta_\mu(0) \to \theta(0) \quad \text{in} \quad H_0^1(\Omega), \\ \theta'_\mu(0) \to \theta'(0) \quad \text{in} \quad L^2(\Omega), \\ \dfrac{\partial \theta_\mu}{\partial \nu} \to \dfrac{\partial \theta}{\partial \nu} \quad \text{in} \quad L^2(\Sigma). \end{array}\right. \tag{4.26}$$

But z satisfies the regularity condition (4.15), then $\Delta z \in C^0([0,T]; H^{-1}(\Omega))$. But z is weak solution of (4.1) with initial data (4.19), then $z'' - \Delta z \in C^0([0,T]; H^{-1}(\Omega))$. It follows that makes sense $\langle z'' - \Delta z, \theta_\mu \rangle$ or $\langle z'', \theta_\mu \rangle$, $\langle -\Delta z, \theta_\mu \rangle$, duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Then, by the regularity (4.25) we can use the integration by parts and the argument used to obtain equality (4.2), but now not formally. Then we have:

$$\int_Q z f_\mu \, dxdt = -(z^0, \theta'_\mu(0)) + \langle z^1, \theta_\mu(0) \rangle - \int_\Sigma \frac{\partial \theta_\mu}{\partial \nu} \, v \, d\Gamma dt. \tag{4.27}$$

Taking the limit in (4.27) when $\mu \to \infty$, observing the convergences (4.26), it follows that z is an ultra weak solution of the non homogeneous problem (4.1) with regular initial data given by (4.19). ■

Let us prove now that the ultra weak solution of the non homogeneous mixed problem (4.1) is in $C^0([0,T]; L^2(\Omega))$. In fact, given $z^0 \in L^2(\Omega)$, $z^1 \in H^{-1}(\Omega)$ and $v \in L^2(\Sigma)$, exists sequences $(z_\mu^0)_{\mu \in \mathbb{N}}$, $(z_\mu^1)_{\mu \in \mathbb{N}}$ and $(v_\mu)_{\mu \in \mathbb{N}}$ with z_μ^0, z_μ^1 and v_μ, respectively, in $H_0^1(\Omega)$, $L^2(\Omega)$ and $H_0^2(0,T; H^{3/2}(\Gamma))$ such that:

$$\left|\begin{array}{lll} z_\mu^0 & \text{converges to } z^0 & \text{in } L^2(\Omega), \\ z_\mu^1 & \text{converges to } z^1 & \text{in } H^{-1}(\Omega), \\ v_\mu & \text{converges to } v & \text{in } L^2(\Sigma). \end{array}\right. \tag{4.28}$$

Let $\widehat{v}_\mu \in H_0^1(0,T;H^2(\Omega))$ such that $\widehat{v}_\mu = v_\mu$ on Σ. Consider the non homogeneous mixed problem:

$$\left| \begin{aligned} & z''_\mu - \Delta z_\mu = 0 \quad \text{in} \quad Q, \\ & z_\mu = v_\mu \quad \text{on} \quad \Sigma, \\ & z_\mu(0) = z^0_\mu, \; z'_\mu(0) = z^1_\mu. \end{aligned} \right. \tag{4.29}$$

By Lemma 4.1 it follows that the solution z_μ of (4.29) is in the class:

$$z_\mu \in C^0([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)) \tag{4.30}$$

and z_μ is an ultra weak solution of the problem (4.29). Therefore, if z is ultra weak solution of (4.1), it follows that $z_\mu - z$ is also ultra weak solution of (4.1) with data $z^0_\mu - z^0$, $z^1_\mu - z^1$ and $v_\mu - v$. By the estimate (4.14) of Theorem 4.1, we obtain:

$$||z_\mu - z||_{L^\infty(0,T;L^2(\Omega))} \le C\big(|z^0_\mu - z^0| + ||z^1_\mu - z^1||_{H^{-1}(\Omega)} + ||v_\mu - v||_{L^2(\Sigma)}\big).$$

When $\mu \to \infty$ in the last inequality, we obtain by (4.28),

$$\lim_{\mu\to\infty} z_\mu = z \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)).$$

By $z_\mu \in C^0([0,T];L^2(\Omega))$, then $z \in C^0([0,T];L^2(\Omega))$. ■

Step 2., We prove now that $z' \in C^0([0,T];H^{-1}(\Omega))$. In the proof we use Chapter 3, Lemma 3.3 identity (3.5). We prove first Lemma 4.2 and announce Lemma 4.3 which will be proved latter. Note, however, the notation:

$$W_0^{1,1}(0,T;L^2(\Omega)) = \left\{ v; v, \frac{dv}{dt} \in L^2(0,T;L^2(\Omega)) \text{ and } v(0) = v(T) = 0 \right\},$$

which is a Banach space with the norm:

$$||v||_{W_0^{1,1}(0,T;L^2(\Omega))} = \left\| \frac{dv}{dt} \right\|_{L^1(0,T;L^2(\Omega))}. \tag{4.31}$$

The dual of this Banach space will be represented by $W^{-1,\infty}(0,T;L^2(\Omega))$. For all $f \in W_0^{1,1}(0,T;L^2(\Omega))$, we have:

$$\langle z', f \rangle = -\int_0^T (z, f')\, dt. \tag{4.32}$$

Then, by Schwarz inequality and (4.31), we obtain from (4.32):

$$||z'||_{W^{-1,\infty}(0,T;L^2(\Omega))} \le ||z||_{L^\infty(0,T;L^2(\Omega))}, \tag{4.33}$$

for all weak solution of z of (4.1). ■

Lemma 4.2 *For a weak solution z of (4.1) we have:*

$$z' \in W_0^{-1,\infty}(0,T;L^2(\Omega)).$$

Proof: If z is an weak solution we have $z \in L^\infty(0,T;L^2(\Omega))$, in particular $z \in L^2(0,T;L^2(\Omega))$ what implies $z' \in H^{-1}(0,T;L^2(\Omega))$. Let f in $W^{1,1}(0,T;L^2(\Omega))$ and consider a sequence $(f_\mu)_{\mu\in\mathbb{N}}$ of functions $f_\mu \in H_0^1(0,T;L^2(\Omega))$ such that:

$$f_\mu \to f \quad \text{in} \quad W_0^{1,1}(0,T;L^2(\Omega)). \tag{4.34}$$

We have by (4.32) and (4.33) for f_μ instead of f and taking limit when $\mu \to \infty$, that $z' \in W^{-1,\infty}(0,T;L^2(\Omega))$. ■

Let us consider $f \in H_0^1(0,T;L^2(\Omega))$. From (4.32) and the definition of weak solutions it follows

$$\langle z', f\rangle = -\int_Q z\, f'\, dxdt = (z^0, \theta'(0)) - \langle z^1, \theta(0)\rangle + \int_\Sigma \frac{\partial\theta}{\partial\nu}\, v\, d\Gamma dt \tag{4.35}$$

for all θ solution of the backward problem:

$$\left| \begin{array}{l} \theta'' - \Delta\theta = f' \quad \text{in} \quad Q, \\ \theta = 0 \quad \text{on} \quad \Sigma, \\ \theta(T) = 0,\ \theta'(T) = 0 \quad \text{on} \quad \Omega. \end{array} \right. \tag{4.36}$$

We assume the following lemma which proof will be done later.

Lemma 4.3 *The solution θ of (4.36) satisfies the inequality:*

$$|\theta'(0)| + ||\theta(0)|| + \left\| \frac{\partial\theta}{\partial\nu} \right\|_{L^2(\Sigma)} \leq C||f||_{L^1(0,T;H_0^1(\Omega))} \tag{4.37}$$

for all $f \in W_0^{1,1}(0,T;H_0^1(\Omega))$.

Note that the constant C that appears in (4.37) depends only of T and the vector field $(h_k)_{1\leq k\leq n}$.

From (4.35) and Lemma 4.2, we obtain:

$$|\langle z', f\rangle| \leq C(|z^0| + ||z^1||_{H^{-1}(\Omega)} + ||v||_{L^2(\Sigma)}||f||_{L^1(0,T;H_0^1(\Omega))}\,. \tag{4.38}$$

Since $W_0^{1,1}(0,T;H_0^1(\Omega))$ is dense in $L^1(0,T;H_0^1(\Omega))$, it follows that the inequality (4.38) is true for all $f \in L^1(0,T;H_0^1(\Omega))$, consequently

$$z' \in L^\infty(0,T;H^{-1}(\Omega)) \tag{4.39}$$

$$||z'||_{L^\infty(0,T;H^{-1}(\Omega))} \le C\big(|z^0| + ||z^1||_{H^{-1}(\Omega)} + ||v||_{L^2(\Sigma)}\big)\,. \tag{4.40}$$

Note that (4.39) and (4.40) are verified for all ultra weak solution of the problem (4.1).

Now, let us consider a sequence of weak solutions of (4.1), approximating z as in (4.28). Then $z_\mu - z$ is also an ultra weak solution of (4.1) and by (4.40) we obtain:

$$||z'_\mu - z'||_{L^\infty(0,T;H^{-1}(\Omega))} \le C\big(|z^0_\mu - z^0| + ||z^1_\mu - z^1||_{H^{-1}(\Omega)} + ||v_\mu - v||_{L^2(\Sigma)}\big).$$

Whence

$$\lim_{\mu\to\infty} z'_\mu = z' \quad \text{in} \quad L^\infty(0,T;H^{-1}(\Omega)). \tag{4.41}$$

Note that z_μ is also weak solution, then $z'_\mu \in C^0([0,T];H^{-1}(\Omega))$ and by (4.41) we have $z' \in C^0([0,T];H^{-1}(\Omega))$. ∎

Proof: Let us consider the problem

$$\left|\begin{array}{l} w'' - \Delta w = f \quad \text{in} \quad Q, \\ w = 0 \quad \text{on} \quad \Sigma, \\ w(T) = 0,\ w'(T) = 0 \quad \text{in} \quad \Omega \end{array}\right. \tag{4.42}$$

for $f \in W^{1,1}_0(0,T;H^1_0(\Omega))$. It follows, from the regularity of strong solutions, Chapter 1, Theorem 3.1, that:

$$w \in C^0([0,T];H^1_0(\Omega)\cap H^2(\Omega)) \cap C^1([0,T];H^1_0(\Omega)) \tag{4.43}$$

$$||w'||_{L^\infty(0,T;H^1_0(\Omega))} + ||w||_{L^\infty(0,T;H^1_0(\Omega)\cap H^2(\Omega))} \le C||f||_{L^1(0,T;H^1_0(\Omega))}\,. \tag{4.44}$$

Let $w' = \theta$. Then θ is solution of (4.36) because θ verifies the equation $(4.36)_1$ $\theta(T) = w'(T) = 0$ and $\theta'(T) = w''(T) = \Delta w(T) = 0$, because $f \in W^{1,1}_0(0,T;H^1_0(\Omega))$.

Whence,

$$|\theta'(0)| + ||\theta(0)|| = |w''(0)| + ||w'(0)|| = |\Delta w(0)| + ||w'(0)||.$$

It follows from (4.44) that:

$$|\theta'(0)| + ||\theta(0)|| \le C||f||_{L^1(0,T;H^1_0(\Omega))}\,. \tag{4.45}$$

Note that with (4.45), in order to obtain the inequality of Lemma 4.2, it is sufficient to estimate $\left\|\dfrac{\partial\theta}{\partial\nu}\right\|_{L^2(\Sigma)}$ by $||f||_{L^1(0,T;H^1_0(\Omega))}$. For this, we use the identity (3.5) of Lemma 3.3, Chapter 3. In fact, we rewrite it for θ solution of (4.36). We have with $q_k = h_k$:

$$\begin{aligned} &\frac{1}{2}\int_\Sigma \left(\frac{\partial\theta}{\partial\nu}\right)^2 d\Gamma dt = -\left(\theta(0), h_k\,\frac{\partial\theta(0)}{\partial x_k}\right) + \\ &+\frac{1}{2}\int_Q \frac{\partial h_k}{\partial x_k}\,(|\theta'|^2 - ||\theta||^2)\,dxdt + \int_Q \frac{\partial h_k}{\partial x_j}\,\frac{\partial\theta}{\partial x_k}\,\frac{\partial\theta}{\partial x_j}\,dxdt - \\ &-\int_Q f'\,h_k\,\frac{\partial\theta}{\partial x_k}\,dxdt. \end{aligned} \tag{4.46}$$

Since $h_k \dfrac{\partial \theta}{\partial x_k} \in L^\infty(0,T;L^2(\Omega))$ it follows that $h_k \dfrac{\partial \theta'}{\partial x_k} \in W^{-1,\infty}(0,T;L^2(\Omega))$, then

$$-\int_Q f'\, h_k \frac{\partial \theta}{\partial x_k}\, dxdt = \int_Q f\, h_k \frac{\partial \theta'}{\partial x_k}\, dxdt. \tag{4.47}$$

Also, as f is zero in T and $\theta' = w'' = \Delta w + f$, we obtain:

$$\begin{aligned} \int_Q f\, h_k \frac{\partial \theta'}{\partial x_k}\, dxdt &= -\int_Q \frac{\partial}{\partial x_k}(f\, h_k)\theta'\, dxdt \; = \\ = -\int_Q \frac{\partial f}{\partial x_k} h_k \Delta w\, dxdt &- \int_Q \frac{\partial f}{\partial x_k} h_k\, f\, dxdt - \\ -\int_Q \frac{\partial h_k}{\partial x_k} f \Delta w\, dxdt &- \int_Q \frac{\partial h_k}{\partial x_k} f^2\, dxdt. \end{aligned} \tag{4.48}$$

By the same argument, we have:

$$-\int_Q \left(\frac{\partial f}{\partial x_k}\right) h_k\, f\, dxdt = -\frac{1}{2}\int_Q h_k \frac{\partial}{\partial x_k} f^2\, dxdt = \frac{1}{2}\int_Q \frac{\partial h_k}{\partial x_k} f^2\, dxdt. \tag{4.49}$$

Substituting (4.49) in (4.48) and the result in (4.47), we obtain:

$$\begin{aligned} -\int_Q f' h_k \frac{\partial \theta}{\partial x_k}\, dxdt &= -\int_Q \left(\frac{\partial f}{\partial x_k}\right) h_k \Delta w\, dxdt - \\ -\int_q \left(\frac{\partial h_k}{\partial x_k}\right) f \Delta w\, dxdt &- \frac{1}{2}\left(\frac{\partial h_k}{\partial x_k}\right) f^2\, dxdt. \end{aligned} \tag{4.50}$$

We know that:

$$\begin{aligned} &\frac{1}{2}\int_Q \frac{\partial h_k}{\partial x_k}\left(|\theta'|^2 - ||\theta||^2\right) dxdt \; = \\ &= \frac{1}{2}\int_Q \frac{\partial h_k}{\partial x_k}\left(|\Delta w|^2 + 2f|\Delta w| + |f|^2 - ||\theta||^2\right) dxdt. \end{aligned} \tag{4.51}$$

Substituting (4.50) and (4.51) in (4.46) we have:

$$\begin{aligned} &\frac{1}{2}\int_\Sigma \left(\frac{\partial \theta}{\partial \nu}\right)^2 d\Gamma dt = -\left(w'(0), h_k \frac{\partial w'(0)}{\partial x_k}\right) + \\ &+\frac{1}{2}\int_Q \left(\frac{\partial h_k}{\partial x_k}\right) |\Delta w|^2\, dxdt - \frac{1}{2}\int_Q \left(\frac{\partial h_k}{\partial x_k}\right) ||w'||^2\, dxdt - \\ &-\int_Q \left(\frac{\partial f}{\partial x_k}\right) h_k \Delta w\, dxdt + \int_Q \frac{\partial h_k}{\partial x_j}\frac{\partial w'}{\partial x_k}\frac{\partial w'}{\partial x_j}\, dxdt. \end{aligned} \tag{4.52}$$

Applying the estimate (4.44) to the right hand side of (4.52), observing that $h_k \in C^1(\overline{\Omega})$, $1 \le k \le n$, we obtain:

$$\int_\Sigma \left(\frac{\partial \theta}{\partial \nu}\right)^2 d\Gamma dt \le C||f||_{L^1(0,T;H_0^1(\Omega))}\,. \tag{4.53}$$

From (4.45) and (4.53) follows the proof of Lemma 4.3. ■

The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.2 *The linear mapping* $\{z^0, z^1, v\} \to \{z, z'\}$ *from* $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma)$ *into* $L^\infty(0,T; L^2(\Omega)) \times L^\infty(0,T; H^{-1}(\Omega))$*, where* z *is the ultra weak solution of (4.1), is continuous.*

Chapter 5

Concrete Representation of Ultra Weak Solutions

5.1 Concrete Representation of Ultra Weak Solutions

The most difficult point in this section is to prove that the ultra weak solution z, Chapter 4, (4.1), has trace on the lateral boundary Σ of the cylinder $Q = \Omega \times]0, T[$. To make it clear we need an appropriate trace operator what will be done in the following.

Let us consider the Hilbert space

$$U = \left\{ u \in L^2(\Omega); \Delta u \in H^{-1}(\Omega) \right\}$$

with the norm:

$$||u||_U^2 = |u|^2 + ||\Delta u||_{H^{-1}(\Omega)}^2 .$$

Following the argument of Lions [32], we prove that $\mathcal{D}(\overline{\Omega})$ is dense in U. Note that by $\mathcal{D}(\overline{\Omega})$ we represent the restrictions to Ω of the functions φ of $C^\infty(\mathbb{R}^n)$. Then, if $u \in U$ it has a trace on Γ, that is, we can construct a continuous linear operator γ_0 such that:

$$u \in U \to \gamma_0 u \in H^{-\frac{1}{2}}(\Gamma), \tag{5.1}$$

such that $\gamma_0 \varphi = \varphi|_\Gamma$ for all $\varphi \in \mathcal{D}(\overline{\Omega})$.

Let V be the Hilbert space

$$V = \left\{ v \in L^2(0, T; L^2(\Omega));\ \Delta v \in L^2(0, T; H^{-1}(\Omega)) \right\},$$

with the norm:

$$||v||_V^2 = ||v||_{L^2(0,T;L^2(\Omega))}^2 + ||\Delta v||_{L^2(0,T;H^{-1}(\Omega))}^2 .$$

Using the density of $\mathcal{D}(\overline{\Omega})$ in U, we obtain, directly, that the set

$$\{\eta\varphi;\ \eta \in C_0^\infty(0,T), \varphi \in \mathcal{D}(\overline{\Omega})\}$$

is total in V. Using (5.1) we define γ_0 for functions of V. To simplify the notation we use the same in (5.1), that is, we write:

$$(\gamma_0 v)(t) = \gamma_0 v(t) \quad \text{for all} \quad t \in]0,T[\,.$$

It follows from (5.1) that $\gamma_0 v \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$ and

$$\gamma_0 \colon V \to L^2(0,T;H^{-\frac{1}{2}}(\Gamma)) \tag{5.2}$$

is linear and continuous.

Let us consider the Hilbert space $H_0^1(0,T;H^{-\frac{1}{2}}(\Gamma))$ which is the space of $w \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$ such that $\dfrac{dw}{dt} \in L^2(0,T;H^{-\frac{1}{2}}(\Gamma))$ with $w(0) = w(T) = 0$. The norm in this space is:

$$||w||_{H_0^1(0,T;H^{-\frac{1}{2}}(\Gamma))} = \left\|\frac{dw}{dt}\right\|_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma))}.$$

The dual space of $H_0^1(0,T;H^{-\frac{1}{2}}(\Gamma))$ is represented by $H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))$.

For the functions $v \in V$ we define the map $\widetilde{\gamma}_0$ in the following manner:

$$\langle \widetilde{\gamma}_0 v', w\rangle = -\int_0^T (\gamma_0 v, w')_{H^{-\frac{1}{2}}(\Gamma)\times H^{-\frac{1}{2}}(\Gamma)}\, dt \tag{5.3}$$

for all $w \in H_0^1(0,T;H^{-\frac{1}{2}}(\Gamma))$.

From (5.3) we obtain:

$$|\langle \widetilde{\gamma}_0 v', w\rangle| \le ||\gamma_0 v||_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma))} \cdot ||w'||_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma))}.$$

By definition of norm in $H_0^1(0,T;H^{-\frac{1}{2}}(\Gamma))$ we obtain:

$$|\langle \widetilde{\gamma}_0 v', w\rangle| \le ||\gamma_0 v||_{L^2(0,T;H^{-1}(\Gamma))} \cdot ||w||_{H_0^1(0,T;H^{-\frac{1}{2}}(\Gamma))}. \tag{5.4}$$

or

$$||\widetilde{\gamma}_0 v'||_{H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))} \le C||v||_V\,.$$

We observe that if $v = \eta\varphi$, $\eta \in C_0^\infty(0,T)$ and $\varphi \in \mathcal{D}(\overline{\Omega})$, then:

$$\begin{aligned}\langle \widetilde{\gamma}_0 v', w\rangle &= -\int_0^T (\gamma_0(\eta\varphi), w')\, dt = -\int_0^T \eta(\gamma_0\varphi, w')_{L^2(\Gamma)}\, dt = \\ &= \int_0^T \eta'(\gamma_0\varphi, w)_{L^2(\Gamma)}\, dt = \langle \gamma_0(\eta'\varphi), w\rangle,\end{aligned}$$

that is, $\widetilde{\gamma}_0 v' = \gamma_0 v'$. We have proved the following:

Theorem 5.1 *The map* $\widetilde{\gamma}_0$ *defined by (5.3) takes values in* $H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))$ *and*

$$\widetilde{\gamma}_0\colon V \to H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))$$

is linear and continuous.

Proof: As we have seen $\widetilde{\gamma}_0 v = \gamma_0 v'$ for $v = \eta\varphi$, with $\eta \in C_0^\infty(0,T)$, $\varphi \in \mathcal{D}(\overline{\Omega})$, the map $\widetilde{\gamma}_0$ is called trace application for functions v' with $v \in V$. ■

Now let us return to the study of trace on Σ for ultra weak solution z of Chapter 4, (4.1).

In fact, let be $\theta \in C_0^\infty(Q)$. Then θ is solution of the problem (4.5), Chapter 4, with $f = \theta'' - \Delta\theta$. Substituting this f in the expression (4.15), Chapter 4, we obtain:

$$\int_Q z(\theta'' - \Delta\theta)\, dxdt = 0,$$

because $\theta \in C_0^\infty(Q)$. Whence

$$\langle z'' - \Delta z, \theta\rangle = 0 \quad \text{for all} \quad \theta \in C_0^\infty(Q).$$

Consequently,

$$z'' - \Delta z = 0 \quad \text{a.e. in} \quad Q. \tag{5.5}$$

As a consequence of (5.5), since $z \in C^0([0,T];L^2(\Omega))$, it follows that:

$$z'' \in C^0([0,T];H^{-2}(\Omega)). \tag{5.6}$$

Note that $\Delta\colon L^2(\Omega) \to H^{-1}(\Omega)$ is linear and continuous.

Let us consider $\theta = \eta\varphi$, with $\eta \in H^2(0,T)$, $\theta(T) = \theta'(T) = 0$ and $\varphi \in H_0^2(\Omega)$. By (4.15) Chapter 4, definition of weak solution z of (4.1) Chapter 4, we have:

$$\int_Q z(\eta''\varphi - \eta\Delta\varphi)\, dxdt = -(z^0, \eta'(0)\varphi) + \langle z^1, \eta(0)\varphi\rangle. \tag{5.7}$$

Integrating by parts twice with respect to t, applying Green's identity and by regularity of z given by (4.15) Chapter 4, we get:

$$\begin{aligned} \int_Q z(\eta''\varphi - \eta\Delta\varphi)\, dxdt &= -(z(0), \eta'(0)\varphi) + \\ &+ \langle (z'(0), \eta(0)\varphi\rangle + \textstyle\int_0^T \langle z'' - \Delta z, \eta\varphi\rangle\, dt. \end{aligned} \tag{5.8}$$

It follows from (5.7), (5.8) and (5.5), that:

$$-(z^0, \eta'(0)\varphi) + \langle z^1, \eta(0)\varphi\rangle = (-z(0), \eta'(0)\varphi) + \langle z'(0), \eta(0)\varphi\rangle.$$

Choosing conveniently $\eta(0)$ and $\eta'(0)$, we obtain

$$z(0) = z^0, \quad z'(0) = z^1. \tag{5.9}$$

■

Let us prove now that $\widetilde{\gamma}_0 z = v$. In fact, we define:

$$y(t) = \int_0^t z(s)\, ds,$$

where z is the ultra weak solution. Then $y \in L^2(0,T;L^2(\Omega))$ and from the equation (5.5) we get:

$$\Delta y(t) = \Delta \int_0^t z(s)\, ds = \int_0^t \Delta z(s)\, ds = \int_0^t z''(s)\, ds = z'(t) - z'(0). \tag{5.10}$$

From regularity we have $z' \in C^0([0,T];H^{-1}(\Omega))$ and it follows that $\Delta y \in L^2(0,T;H^{-1}(\Omega))$. Thus, $y \in V$ and by Theorem 5.1, we have:

$$\widetilde{\gamma}_0 y' = \widetilde{\gamma}_0 z. \tag{5.11}$$

Let $(z_\mu)_{\mu\in\mathbb{N}}$ be a sequence of solutions of the problem (4.29) Chapter 4. We obtain:

$$\left| \begin{array}{lll} z_\mu \to z & \text{in} & C^0([0,T];L^2(\Omega)) \\ z'_\mu \to z' & \text{in} & C^0([0,T];H^{-1}(\Omega)) \end{array} \right. \tag{5.12}$$

Let us consider

$$y_\mu(t) = \int_0^t z_\mu(s)\, ds.$$

Then

$$\Delta y_\mu = z'_\mu(t) - z'_\mu(0).$$

By convergences (5.12) applied to y_μ, we get:

$$y_\mu \to y \quad \text{in} \quad V.$$

Whence by Theorem 5.1 and definition (5.3) of $\widetilde{\gamma}_0$, we obtain:

$$\widetilde{\gamma}_0 y'_\mu = \widetilde{\gamma}_0 z_\mu \to \widetilde{\gamma}_0 y' = \widetilde{\gamma}_0 z \quad \text{in} \quad H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma)). \tag{5.13}$$

We know that $z_\mu \in C^1([0,T];H^1(\Omega))$, then we get $\gamma_0 z_\mu \in C^1([0,T];H^{1/2}(\Gamma))$ and

$$\gamma_0 \int_0^t z_\mu(s)\, ds = \int_0^t (\gamma_0 z_\mu)(s)\, ds.$$

We have then,

$$(\widetilde{\gamma}_0 y_\mu, w) = -\int_0^T (\gamma_0 y_\mu, w')\,dt \;=$$
$$= -\int_0^T \left(\int_0^t (\gamma_0 z_\mu)(s)\,ds, w'\right) dt = \int_0^T (\gamma_0 z_\mu, w)\,dt,$$

that is,

$$\widetilde{\gamma}_0 z_\mu = \widetilde{\gamma}_0 y'_\mu = \gamma_0 z_\mu\,. \tag{5.14}$$

Observe that

$$\gamma_0 z_\mu = v_\mu \quad \text{and} \quad v_\mu \to v \quad \text{in} \quad L^2(\Sigma). \tag{5.15}$$

As $L^2(\Sigma) = L^2(0,T;L^2(\Gamma)) \subset H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))$ continuously, it follows from (5.13), (5.14) and (5.15) that

$$\widetilde{\gamma}_0 z = v. \tag{5.16}$$

Scholium. The ultra weak solution z of (4.1) Chapter 4, was defined by transposition method at (4.15) Chapter 4. The existence was proved by Riesz's representation of continuous linear form on $L^1(0,T;L^2(\Omega))$. Then the ultra weak solution z is identified to an object of $L^\infty(0,T;L^2(\Omega))$ the dual of $L^1(0,T;L^2(\Omega))$. After it was proved the regularity, that is, $z \in C^0([0,T];L^2(\Omega)) \cap C^1([0,T];H^{-1}(\Omega))$, as shown in Theorem 4.2, Chapter 4. In the present section we proved that the ultra weak solution is a genuine solution, that is, $z'' - \Delta z = 0$, a.e. in Q, cf. (5.5). $z(0) = z^0$, $z'(0) = z^1$ as in (5.5) and the boundary condition $z = v$ on Σ, cf. (5.16). ■

Chapter 6

Boundary Exact Controllability

6.1 Boundary Exact Controllability

We will give, first, a general formulation of HUM (Hilbert Uniqueness Method), idealized by J.L. Lions [36] and [38] or [39]. We begin with an action on the boundary Σ of the cylinder $Q = \Omega \times]0, T[$, where Ω is a bounded open set of $\mathbb{R}^n$ with boundary Γ and $T > 0$ is a real number.

Let us consider the wave equation

$$y'' - \Delta y = 0 \quad \text{in} \quad Q \tag{6.1}$$

with initial condition

$$y(0) = y^0,\ y'(0) = y^1 \quad \text{in} \quad \Omega \tag{6.2}$$

and boundary condition

$$y = v \quad \text{on} \quad \Sigma = \Gamma \times]0, T[\,. \tag{6.3}$$

Physically we can think that the above linear non homogeneous boundary value problem describe the vibrations of an elastic structure Ω of $\mathbb{R}^3$, when the action on the system is done along the boundary Σ. It is interesting to observe that in the applications the action is only on a part Σ_0 of Σ.

Observe that the function $y = y(x, t)$, solution of (6.1), (6.2) and (6.3) depends of $x \in \Omega$, $t \in [0, T[$ and v belongs to a certain space called space of controls. The function v itself is defined as the control function. To make explicit this dependence we write for the solution

$$y = y(x, t, v),\ y = y(v),\ y = y(x, t) \quad \text{or} \quad y = y(t). \tag{6.4}$$

Problem of Exact Controllability. Given $T > 0$, find a Hilbert space H such that for every pair of initial data $\{y^0, y^1\} \in H$, there exists a control v in the set of controls such that the solution $y = y(x,t,v)$ of (6.1), (6.2) and (6.3) verifies the equilibrium condition:

$$y(x,T,v) = 0 \quad \text{and} \quad y'(x,T,v) = 0 \tag{6.5}$$

for all $x \in \Omega$ or $y(T) = 0$, $y'(T) = 0$ for all $x \in \Omega$.

Let us consider a part Σ_0 of Σ, with positive measure, such that $\Sigma_0 \cap \Sigma$ is empty and consider the action of the following type:

$$y = \left| \begin{array}{lll} v & \text{on} & \Sigma_0 \\ 0 & \text{on} & \Sigma \backslash \Sigma_0 \end{array} \right. \tag{6.6}$$

The problem of exact controllability can be formulated as follows: given $T > 0$ find a Hilbert space H such that for every pair of initial data $\{y^0, y^1\}$ in H there exists a control v belonging to the space of controls, defined on Σ_0, such that the solution $y = y(x,t,v)$ of (6.1), (6.2) and (6.6) verifies the equilibrium condition (6.5).

6.2 Description of HUM

The methodology of HUM is based on certain criterium of uniqueness and the construction of a Hilbert H space, by completeness. The method takes in consideration the uniqueness and regularity for solutions of the wave equation as developed in the Chapters 1, 2, 3. We will describe it by steps.

Step 1. Given $\{\phi^0, \phi^1\}$ in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, let us consider the homogeneous boundary value problem

$$\left| \begin{array}{l} \phi'' - \Delta\phi = 0 \quad \text{in} \quad Q, \\ \phi = 0 \quad \text{on} \quad \Sigma, \\ \phi(0) = \phi^0, \ \phi'(0) = \phi^1 \quad \text{in} \quad \Omega. \end{array} \right. \tag{6.7}$$

We know, Chapter 1, that (6.7) has strong solution. By Chapter 3 we obtain

$$\frac{\partial\phi}{\partial\nu} \in L^2(\Sigma). \tag{6.8}$$

Step 2. We solve the backward non homogeneous problem:

$$\left| \begin{array}{l} \psi'' - \Delta\phi = 0 \quad \text{in} \quad Q, \\ \psi = \begin{cases} \dfrac{\partial\phi}{\partial\nu} & \text{on} \ \ \Sigma_0, \\ 0 \ \text{on} \ \ \Sigma\backslash\Sigma_0, \end{cases} \\ \psi(T) = 0, \ \psi'(T) = 0 \quad \text{in} \quad \Omega. \end{array} \right. \tag{6.9}$$

Remark 6.1 *Note that (6.9) is a non homogeneous boundary value problem of the type studied in Chapter 4. To obtain, from (6.9), the system (4.1) of Chapter 4, it is sufficient to consider the change of variable $T-t$ in place of t. Then $\psi(T-t)$ is solution of (6.1), with $y^0 = y^1 = 0$ on Ω. Note that $v = \dfrac{\partial \phi}{\partial \nu}$ is in $L^2(\Sigma)$ by (6.8). We are in the situation of Chapter 4. Consequently (6.9) is a well posed problem. By the regularity obtained in Chapter 4, we can calculate $\psi(0) \in L^2(\Omega)$ and $\psi'(0) \in H^{-1}(\Omega)$.*

The Operator Λ. From the solution ψ of (6.9), we define the application:

$$\Lambda\{\phi^0, \phi^1\} = \{\psi'(0), -\psi(0)\}.$$

Note that from $\{\phi^0, \phi^1\}$ in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ we obtain the solution $\phi = \phi(x,t)$ of (6.7) with regularity (6.8) for the normal derivative. Then, the problem (6.9) is well posed, from which we define Λ. Thus Λ is well defined.

Step 3. Multiply both sides of equation $(6.9)_1$ by $\phi = \phi(x,t)$ solution of (6.7) and integrate in Q. We obtain:

$$\int_Q \psi''\phi \, dxdt - \int_Q \Delta\psi\phi \, dxdt = 0.$$

Analysis of the first integral – We have:

$$(\psi', \phi)' = (\psi'', \phi) + (\psi', \phi').$$

Integrating on $]0, T[$ we obtain:

$$\langle \psi'(T), \phi(T) \rangle - \langle \psi'(0), \phi(0) \rangle = \int_Q \psi''\phi \, dxdt + \int_0^T (\psi', \phi') \, dt.$$

By condition $(6.9)_3$ we modify the last equality obtaining

$$\int_Q \psi''\phi \, dxdt = -\langle \psi'(0), \phi^0 \rangle - \int_0^T (\psi', \phi') \, dt. \tag{6.10}$$

By the same argument we modify the integral on the right hand side of (6.10), obtaining:

$$\int_0^T (\psi', \phi') \, dt = -(\psi(0), \phi^1) - \int_Q \psi\phi'' \, dxdt. \tag{6.11}$$

Substituting (6.11) in (6.10) we obtain:

$$\int_Q \psi''\phi \, dxdt = -\langle \psi'(0), \phi^0 \rangle + \langle \psi(0), \phi^1 \rangle + \int_Q \psi\phi'' \, dxdt. \tag{6.12}$$

Analysis of the second integral – Integrating by parts, we have:

$$-\int_Q \Delta\psi\phi\, dxdt = \int_Q \nabla\psi\cdot\nabla\phi\, dxdt - \int_\Sigma \frac{\partial\psi}{\partial\nu}\,\phi\, d\Gamma\, dt$$

and

$$-\int_Q \Delta\phi\psi\, dxdt = \int_Q \nabla\phi\cdot\nabla\psi\, dxdt - \int_\Sigma \frac{\partial\phi}{\partial\nu}\,\psi\, d\Sigma.$$

Then

$$-\int_Q \Delta\psi\phi\, dxdt = -\int_Q \Delta\phi\psi\, dxdt + \int_\Sigma \frac{\partial\phi}{\partial\nu}\,\psi\, d\Gamma\, dt. \tag{6.13}$$

Adding (6.12) and (6.13), since $\phi'' - \Delta\phi = 0$ a.e. in Q and also $\psi'' - \Delta\psi = 0$ a.e. in Q, by Chapter 5, we obtain:

$$-\langle\psi'(0),\phi^0\rangle + (\psi(0),\phi^1) + \int_\Sigma \frac{\partial\phi}{\partial\nu}\,\psi\, d\Gamma\, dt = 0. \tag{6.14}$$

Note that $\psi = \dfrac{\partial\phi}{\partial\nu}$ on Σ_0 and $\psi = 0$ on $\Sigma\backslash\Sigma^0$. Then, from (6.14) we obtain:

$$-(\psi(0),\phi^1) + \langle\psi'(0),\phi^0\rangle = \int_\Sigma \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma\, dt. \tag{6.15}$$

Consider the first hand side of (6.15) as the inner product of $\{\psi'(0), -\psi(0)\}$ with $\{\phi^0,\phi^1\}$. We then obtain from (6.15):

$$\begin{aligned}&\langle\Lambda\{\phi^0,\phi^1\},\{\phi^0,\phi^1\}\rangle = -(\psi(0),\phi^1)+\\ &+\langle\psi'(0),\phi^1\rangle = \int_\Sigma \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma\, dt.\end{aligned} \tag{6.16}$$

We define in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ the quadratic form:

$$||\{\phi^0,\phi^1\}||_F = \left(\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma\, dt\right)^{1/2} \tag{6.17}$$

which is a semi norm in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$. To obtain a norm, from (6.17), we need to prove: if ϕ is a solution of (6.7) with $\{\phi^0,\phi^1\}\in\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$, then if $\dfrac{\partial\phi}{\partial\nu} = 0$ on Σ_0 this implies ϕ is zero in Q. This is true by Holmgren's theorem, cf. Hörmander [23] and Lions [39]. Then, by Holmgren's theorem it follows that the quadratic form (6.17) is, in fact, a norm in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$.

Remark 6.2 *The norm (6.17) induces in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ the following inner product:*

$$\big\langle\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\big\rangle_F = \int_\Sigma \frac{\partial\phi}{\partial\nu}\frac{\partial\zeta}{\partial\nu}\, d\Gamma\, dt,$$

where $\zeta = \zeta(x,t)$ is the solution of (6.7) corresponding to the initial data $\{\zeta^0,\zeta^1\}\in\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$.

It follows from Remark 6.2 and (6.16) that:

$$\left\langle \Lambda\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\right\rangle = \left\langle \{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\right\rangle_F . \tag{6.18}$$

By Schwarz inequality we obtain:

$$\left|\left\langle \Lambda\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\right\rangle\right| \le \left\|\{\phi^0,\phi^1\}\right\|_F \left\|\{\zeta^0,\zeta^1\}\right\|_F ,$$

proving the continuity of the bilinear form defined by Λ in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$. Let us consider the completion of $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ with respect to $\|\{\phi^0,\phi^1\}\|_F$ defined by (6.17) and represent by F this Hilbert space. The continuous bilinear form $\left\{\{\phi^0,\phi^1\},\{\psi^0,\psi^1\}\right\} \to \left\langle \Lambda\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\right\rangle$ has an extension, by continuity, to the closure F. We continue representing this extension with the same notation. Then, we obtain a continuous bilinear form on the Hilbert space F which is coercive, by Remark 6.2. Then, by Lax-Milgram's Lemma, for each $\{\eta^0,\eta^1\}\in F'$, dual of F, exists a unique $\{\phi^0,\phi^1\}\in F$ such that

$$\left\langle \Lambda\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\right\rangle = \left\langle \{\eta^0,\eta^1\},\{\zeta^0,\zeta^1\}\right\rangle_{F'\times F} \tag{6.19}$$

for all $\{\zeta^0,\zeta^1\}\in F$. Then, for each $\{\eta^0,\eta^1\}\in F'$ exists a unique $\{\phi^0,\phi^1\}\in F$ which is solution of the equation $\Lambda\{\phi^0,\phi^1\}=\{\eta^0,\eta^1\}$ in F'. In fact $\Lambda\colon F\to F'$ is an isomorphism.

Consequently, for each $\{y^1,y^0\}\in F'$ exists a unique $\{\phi^0,\phi^1\}\in F$ such that

$$\Lambda\{\phi^0,\phi^1\}=\{y^1,-y^0\} \quad \text{in} \quad F'.$$

Note that the map Λ was defined by $\Lambda\{\phi^0,\phi^1\}=\{\psi'(0),-\psi(0)\}$, where ψ is the solution of the non homogeneous problem (6.9). Therefore,

$$\psi'(0)=y^1 \quad \text{and} \quad \psi(0)=y^0.$$

Thus, considering $\tau=T-t$ instead of t in (6.9) and the control $v=\dfrac{\partial\phi}{\partial\nu}$ for (6.1), (6.2) and (6.6), we have $\psi=\psi(x,t)$ and $y=y(x,t)$ are ultra weak solutions of the same non homogeneous boundary value problem. By uniqueness of ultra weak solutions, it follows that $y(x,t)=\psi(x,t)$ for all (x,t) in Q. Therefore by $(6.9)_3$ it follows that:

$$y(x,T)=0 \quad \text{and} \quad y'(x,T)=0 \quad \text{in} \quad \Omega$$

which is the condition (6.5). ■

Remark 6.3 *Note that the operator Λ is symmetric, look Remark 6.2, for example. Then, the solution $\{\phi^0,\phi^1\}$ of the equation $\Lambda\{\phi^0,\phi^1\}=\{y^1,-y^0\}$ can be obtained by a minimization process. In fact, $\{\phi^0,\phi^1\}$ is obtained by:*

$$\underset{\{\zeta^0,\zeta^1\}\in F}{Min}\left\{\frac{1}{2}\left\langle\Lambda\{\zeta^0,\zeta^0\},\{\zeta^0,\zeta^1\}\right\rangle - \left\langle\{y^1,-y^0\},\{\zeta^0,\zeta^1\}\right\rangle\right\}.$$

For numerical analysis, cf. Glowinski, Li, Lions [19].

The next step is to characterize the spaces F and F' as Sobolev spaces. In fact, we know by Chapter 3 that the weak solution $\phi = \phi(x,t)$ of (6.7) satisfies the inequality:

$$\int_\Sigma \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma\, dt \le C_0\, E(0) = C_0 \left\|\{\phi^0, \phi^1\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} . \tag{6.20}$$

Since F is the completion with the norm defined by the right hand side of (6.20), we obtain $H_0^1(\Omega)\times L^2(\Omega) \subset F$. In order to prove that $F \subset H_0^1(\Omega)\times L^2(\Omega)$ we need to prove that exists C_1 such that

$$C_1 \left\|\{\phi^0, \phi^1\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \le \int_\Sigma \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma\, dt. \tag{6.21}$$

If we prove (6.21) it follows that $F = H_0^1(\Omega)\times L^2(\Omega)$ and its dual is $F' = H^{-1}(\Omega)\times L^2(\Omega)$, consequently everything is in order.

The inequality (6.20) is called **direct** and (6.21) is the **inverse**. To complete HUM for the case of action on the boundary we need only to prove (6.21).

■

6.3 Inverse Inequality.

First let us fix some notations. With Ω we denote a bounded open set of $\mathbb{R}^n$ and x^0 any point of $\mathbb{R}^n$. represent by $m(x)$ the vector $x - x^0$ with components $m_k(x) = x_k - x_k^0$, $1 \le k \le n$. If Γ is the boundary of Ω, we define:

$$\Gamma(x_0) = \{x \in \Gamma; m(x)\cdot\nu(x) > 0\} \quad \text{and} \quad \Gamma_*(x^0) = \{x \in \Gamma; m(x)\cdot\nu(x) \le 0\}.$$

$$\Sigma(x^0) = \Gamma(x_0)\times]0,T[\quad \text{and} \quad \Sigma_*(x^0) = \Gamma_*(x^0)\times]0,T[\,.$$

$$R(x^0) = \sup_{x\in\overline{\Omega}} ||x - x^0|| = ||m(x)||_{L^\infty(\Omega)}\,.$$

Theorem 6.1 *Consider $T(x^0) = 2R(x^0)$. If $T > T(x^0)$ then:*

$$\left\|\phi^0\right\|^2_{H_0^1(\Omega)} + \left|\phi^1\right|^2_{L^2(\Omega)} \le \frac{R(x^0)}{T - T(x^0)} \int_{\Sigma(x^0)} \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma\, dt, \tag{6.22}$$

for all weak solutions $\phi = \phi(x,t)$ of (6.7).

Proof: For completeness of the argument, we rewrite the identity (3.5) Chapter 3. In fact, for all vector field $q = (q_k)_{1\le k\le n}$ with $q_k \in C^1(\overline{\Omega})$, $1 \le k \le n$ and all weak solution ϕ of (6.7), we have the identity:

$$\frac{1}{2}\int_\Sigma q_k \cdot \nu_k \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma\, dt = \left(\phi'(t), q_k \frac{\partial \phi}{\partial x_k}\right)\Big|_0^T + \\ + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\left(|\phi'|^2 - |\nabla\phi|^2\right) dxdt + \int_Q \frac{\partial q_k}{\partial x_j}\frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_j}\, dxdt.$$

Choose $q_k(x) = x_k - x_k^0$, $1 \le k \le n$. Then $\sum\limits_{k=1}^n \frac{\partial q_k}{\partial x_k} = n$ and $\frac{\partial q_k}{\partial x_j}\frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_j} = |\nabla \phi|^2$. With this choice for q_k , the above identity changes in the following:

$$X + \frac{n}{2}\int_Q (|\phi'|^2 - |\nabla\phi|^2)\, dxdt + \int_Q |\nabla\phi|^2\, dxdt = \frac{1}{2}\int_\Sigma m_k \nu_k \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma\, dt,$$

where $X = \left(\phi'(t), q_k \frac{\partial \phi}{\partial x_k}\right)\Big|_0^T$.
In $\Sigma(x^0)$ we have:

$$0 \le m_k \cdot \nu_k \le \left(\sum_{k=1}^n m_k^2\right)^{1/2}\left(\sum_{k=1}^n \nu_k^2\right)^{1/2} = ||m(x)|| \le R(x^0).$$

Therefore:

$$\int_\Sigma m_k \nu_k \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma\, dt \le \int_{\Sigma(x^0)} m_k \nu_k \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma\, dt \le R(x^0)\int_{\Sigma(x^0)} \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma\, dt.$$

We obtain:

$$X + \frac{n}{2}\int_Q (|\phi'|^2 - |\nabla\phi|^2)\, dxdt + \int_Q |\nabla\phi|^2\, dxdt \le \frac{R(x^0)}{2}\int_{\Sigma(x^0)} \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma\, dt. \qquad (6.23)$$

To the first hand side of (6.23) add and subtract $\frac{1}{2}\int_Q |\phi'|^2\, dxdt$ and divide $\int_Q |\nabla\phi|^2\, dxdt$ in two parts. Represent $\int_Q (|\phi'|^2 - |\nabla\phi|^2) dxdt$ by Y. Then we obtain from (6.23):

$$X + \frac{n-1}{2} Y + T\, E(0) \le \frac{R(x^0)}{2}\int_{\Sigma(x^0)} \left(\frac{\partial \phi}{\partial \nu}\right)^2 dxdt. \qquad (6.24)$$

Note that the energy $E(t)$ is given by:

$$E(t) = \frac{1}{2}\int_\Omega (|\phi'|^2 + |\nabla\phi|^2)\, dx$$

and we have by energy conservation, that $E(t) = E(0)$.

Lemma 6.1 *For all solution ϕ of the wave equation (6.7) we have:*

$$Y = \int_Q (|\phi'|^2 - |\nabla\phi|^2)\, dxdt = (\phi'(t), \phi(t))\Big|_0^T .$$

Proof: It is sufficient multiply both sides of the equation $(6.7)_1$ by ϕ and integrate by parts. We have:

$$-\int_Q \phi'^2\, dxdt + (\phi'(t), \phi(t))\Big|_0^T + \int_Q |\nabla\phi|^2\, dxdt = 0.$$

■

From Lemma 6.1 we obtain:

$$X + \frac{n-1}{2} Y + \int_\Omega \phi' \left(m \cdot \nabla\phi + \frac{n-1}{2}\, \phi \right) dx\Big|_0^T . \tag{6.25}$$

Consider $\mu > 0$ to be chosen later. We have:

$$\int_\Omega \phi' \left(m \cdot \nabla\phi + \frac{n-1}{2}\, \phi \right) dx \le \frac{\mu}{2} \int_\Omega \phi'^2\, dx + \frac{1}{2\mu} \int_\Omega \left(m \cdot \nabla\phi + \frac{n-1}{2}\, \phi \right)^2 dx. \tag{6.26}$$

We modify the second integral of the right hand of (6.26) as follows:

$$\begin{aligned} &\int_Q \left(m \cdot \nabla\phi + \frac{n-1}{2}\, \phi \right)^2 dx = \int_\Omega (m \cdot \nabla\phi)^2\, dx + \int_\Omega \frac{(n-1)^2}{4}\, \phi^2\, dx + \\ &+(n-1) \int_\Omega (m \cdot \nabla\phi)\phi\, dx. \end{aligned} \tag{6.27}$$

We have:

$$\int_\Omega (m \cdot \nabla\phi)\phi\, dx = \int_\Omega \sum_{k=1}^n m_k \frac{\partial\phi}{\partial x_k}\, \phi\, dx = \frac{1}{2} \int_q \sum_{k=1}^n m_k \frac{\partial}{\partial x_k}\, \phi\, dx. \tag{6.28}$$

By Gauss lemma, since $\phi = 0$ on Σ, we obtain:

$$\int_\Omega \frac{\partial}{\partial x_k} (m_k \phi^2)\, dx = \int_\Gamma \nu_k \cdot m_k\, \phi^2\, d\Gamma = 0$$

$$\frac{1}{2} \int_\Omega \sum_{k=1}^n m_k \frac{\partial}{\partial x_k}\, \phi^2\, dx = -\frac{1}{2} \int_\Omega \sum_{k=1}^n \frac{\partial m_k}{\partial x_k}\, \phi^2\, dx = -\frac{n}{2} \int_\Omega \phi^2\, dx.$$

Substituting in (6.28) and then in (6.27) we obtain:

$$\begin{aligned} &\int_\Omega \left(m \cdot \nabla\phi + \frac{n-1}{2}\, \phi \right)^2 dx = \int_\Omega (m \cdot \nabla\phi)^2\, dx + \\ &+ \left[\frac{(n-1)^2}{4} - \frac{n(n-1)}{2} \right] \int_\Omega \phi^2\, dx \int_\Omega (m \cdot \nabla\phi)^2\, dx. \end{aligned}$$

Note that

$$\int_\Omega (m\cdot\nabla\phi)^2\,dx = \int_\Omega \left(m_k\,\frac{\partial\phi}{\partial x_k}\right)^2 dx \le \int_\Omega (\Sigma m_k)^{1/2}\left(\Sigma\,\frac{\partial\phi}{\partial x_k}\right)^{1/2} dx \;=$$

$$= \int_\Omega ||m(x)||^2\,|\nabla\phi|^2\,dx \le R(x^0)\int_\Omega |\nabla\phi|^2\,dx.$$

Whence,

$$\int_\Omega \left(m\cdot\nabla\phi + \frac{n-1}{2}\,\phi\right)^2 dx \le R(x^0)\int_\Omega |\nabla\phi|^2\,dx. \tag{6.29}$$

Substituting (6.29) in (6.26) and taking $\mu = R(x^0)$ we obtain:

$$\int_\Omega \phi'\left(m\cdot\nabla\phi + \frac{n-1}{2}\,\phi\right)dx \le R(x^0)E(0). \tag{6.30}$$

By (6.26) and (6.30) we obtain:

$$\left|X + \frac{n-1}{2}\,Y\right| = \left|\left(\phi'(t), m\cdot\nabla\phi + \frac{n-1}{2}\,\phi\right)\Big|_0^T\right| \le$$

$$\le 2\left\|\left(\phi'(t), m\cdot\nabla\phi + \frac{n-1}{2}\,\phi\right)\right\|_{L^\infty(0,T)} \le 2R(x^0)E(0).$$

Since $T(x^0) = 2R(x^0)$ we have:

$$\left|X + \frac{n-1}{2}\,Y\right| \le T(x^0)E(0).$$

Whence, by (6.24)

$$-\,T(x^0)E(0) + TE(0) \le -\left|X + \frac{n-1}{2}\,Y\right| \le$$

$$\le X + \frac{n-1}{2}\,Y + T(x^0)E(0) \le \frac{R(x^0)}{2}\int_{\Sigma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma\,dt.$$

■

Chapter 7

Internal Exact Controllability

7.1 Internal Exact Controllability

Let Ω be a bounded open set of $\mathbb{R}^n$ with regular boundary Γ. By w we represent an open subset of Ω and by χ_w we denote the characteristic function of w. Let us consider the boundary value problem:

$$\left| \begin{array}{l} y'' - \Delta y = h\chi_w \quad \text{in} \quad Q \\ y = 0 \quad \text{on} \quad \Sigma \\ y(0) = y^0, \ y'(0) = y^1 \quad \text{in} \quad \Omega \end{array} \right. \tag{7.1}$$

The exact controllability of (7.1) consists in given $T > 0$, find a Hilbert space H such that for every $\{y^0, y^1\} \in H$ exists a control $h \in L^2(w\times]0,T[)$ such that the solution $y = y(x,t)$ of (7.1) satisfies:

$$y(T) = 0 \quad \text{and} \quad y'(T) = 0 \quad \text{in} \quad \Omega. \tag{7.2}$$

This type of problem is called **internal exact controllability** because the action is in the cylinder $w\times]0,T[$ contained in $Q = \Omega\times]0,T[$.

In the following we will prove that HUM is well applied to solve the problem of internal exact controllability. We describe it by steps.

Step 1. Given $\{\phi^0, \phi^1\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ we solve the regular problem:

$$\left| \begin{array}{l} \phi'' - \Delta \phi = 0 \quad \text{in} \quad Q, \\ \phi = 0 \quad \text{on} \quad \Sigma, \\ \phi(0) = \phi^0, \ \phi'(0) = \phi^1 \quad \text{in} \quad \Omega. \end{array} \right. \tag{7.3}$$

This mixed boundary value problem has a regular solution $\phi = \phi(x,t)$.

Step 2. With the solution $\phi = \phi(x,t)$ of (7.3) we solve the backward problem:

$$\left| \begin{array}{l} \psi'' - \Delta\psi = \phi\chi_w \quad \text{in} \quad Q, \\ \psi = 0 \quad \text{on} \quad \Sigma, \\ \psi(T) = 0, \ \psi'(T) = 0 \quad \text{in} \quad \Omega. \end{array} \right. \tag{7.4}$$

The operator Λ. With the solution $\psi = \psi(x,t)$ of (7.4) we define the map Λ by

$$\Lambda\{\phi^0, \phi^1\} = \{\psi'(0), -\psi(0)\}. \tag{7.5}$$

Step 3. Multiply both sides of (7.3) by ψ solution of (7.4) and integrates on Q. We get:

$$\int_0^T\int_\Omega \phi''\psi\, dxdt - \int_0^T\int_\Omega \Delta\phi\psi\, dxdt = 0. \tag{7.6}$$

We obtain, integrating by parts on $]0,T[$ the derivative $(\phi',\psi)' = (\phi'',\psi) + (\phi',\psi')$,

$$(\phi'(T),\psi'(T)) - (\phi'(0),\psi'(0)) = \int_0^T (\phi'',\psi)\, dt + \int_0^T (\phi',\psi')\, dt.$$

Since $\psi(T) = 0$ it follows from the above equality:

$$-(\phi^1, \psi(0)) - \int_0^T (\phi',\psi')\, dt = \int_0^T (\phi'',\psi)\, dt.$$

We also have $(\phi,\psi')' = (\phi',\psi') + (\phi,\psi'')$. By a similar argument we have:

$$-(\phi^0, \psi'(0)) - \int_0^T (\phi,\psi'')\, dt = \int_0^T (\phi',\psi')\, dt.$$

Consequently

$$\int_0^T\int_\Omega \phi''\psi\, dxdt = -(\phi^1,\psi(0)) + (\phi^0,\psi'(0)) + \int_0^T\int_\Omega \phi\psi''\, dxdt. \tag{7.7}$$

Since $\psi = 0$ and $\phi = 0$ on Σ, the Green's formula gives:

$$\int_0^T \int_\Omega \Delta\phi\psi\, dxdt = \int_0^T \int_\Omega \phi\Delta\psi\, dxdt. \tag{7.8}$$

From (7.6), (7.7) and (7.8), noting that ψ is solution of (7.4) we obtain:

$$\int_0^T\int_w \phi^2\, dxdt = (\psi'(0),\phi^0) - (\psi(0),\phi^1). \tag{7.9}$$

By definition (7.5) of Λ we obtain

$$\big\langle \Lambda\{\phi^0,\phi^1\}, \{\phi^0,\phi^1\} \big\rangle = \big(\{\psi'(0), -\psi(0)\}, \{\phi^0,\phi^1\} \big)_F = (\psi'(0),\phi^0) - (\psi(0),\phi^1).$$

Then, by (7.9), we have:

$$\big\langle \Lambda\{\phi^0,\phi^1\},\{\phi^0,\phi^1\}\big\rangle = \int_0^T\!\!\int_w \phi^2\,dxdt. \tag{7.10}$$

Let us define in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ the seminorm

$$\big\|\{\phi^0,\phi^1\}\big\|_F^2 = \int_0^T\!\!\int_w \phi^2\,dxdt. \tag{7.11}$$

Note that to obtain a norm from (7.11) we need to prove that if the solution $\phi=\phi(x,t)$ of (7.3) is zero in $w\times]0,T[$, then $\phi=0$ in Q. The Holmgren's theorem says that there exists $T_0=T_0(w)$, depending of $w\subset\Omega$, such that for every $T>T_0$ the unique solution $\phi=\phi(x,t)$ of (7.3) such that $\phi=0$ on $w\times]0,T[$ is identically zero in Q.

Consequently for $T>T_0$ the quadratic form (7.10) is a norm in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$.

Represent by F the completion of $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ with respect to the norm (7.11), which is a Hilbert space. Note that if $\zeta=\zeta(x,t)$ is the solution of (7.3) corresponding to $\{\zeta^0,\zeta^1\}$ belonging to $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$, then the norm (7.11) is obtained from the inner product in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$ defined by:

$$\big(\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\big)_F = \int_0^T\!\!\int_w \phi\zeta\,dxdt.$$

Let us consider the bilinear form

$$\big\langle \Lambda\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\big\rangle = \int_0^T\!\!\int_w \phi\zeta\,dxdt$$

defined in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$, which is continuous and coercive in $\mathcal{D}(\Omega)\times\mathcal{D}(\Omega)$. Then its extension by continuity to the completion F is also continuous and coercive in F. It follows by Lax-Milgram's lemma, that given $\{y^1,-y^0\}\in F'$, dual of F, there exists a unique $\{\phi^0,\phi^1\}\in F$ such that:

$$\big\langle \Lambda\{\phi^0,\phi^1\},\{\zeta^0,\zeta^1\}\big\rangle = \big\langle\{y^1,-y^0\},\{\zeta^0,\zeta^1\}\big\rangle_{F'\times F}$$

for all $\{\zeta^0,\zeta^1\}\in F$. Then, given $\{y^1,-y^0\}\in F'$, exists $\{\phi^0,\phi^1\}\in F$ such that:

$$\Lambda\{\phi^0,\phi^1\} = \{y^1,-y^0\} \quad \text{in} \quad F'. \tag{7.12}$$

By (7.12) and (7.5) we conclude that:

$$\psi(0)=y^0 \quad \text{and} \quad \psi'(0)=y^1$$

where ψ is the solution of (7.4). In (7.1) we consider h equal to the restriction of ϕ, solution of (7.3), to $w\times]0,T[$. By uniqueness of solution of the linear wave equation we have $\psi(x,t)=y(x,t)$ in Q. Then $y(T)=0$ and $y'(T)=0$, which is (7.2). ■

Observe that the pair $\{\phi^0, \phi^1\}$ is constructive, when we know F, because the bilinear form is symmetric and therefore $\{\phi^0, \phi^1\}$ is obtained by a minimization process as was done in the boundary case, cf. Chapter 6.

The next step is to give a concrete characterization of the completion F. Note that, when we consider $\phi \in L^2(\Omega)$, $\phi^1 \in H^{-1}(\Omega)$, we obtained in Chapter 4, Theorem 4.1, inequality (7.14), for ultra weak solution, which applied to (7.3) gives:

$$\begin{aligned} &\int_0^T\int_w \phi^2\,dxdt \le C_0\big(|\phi^0|^2_{L^2(\Omega)} + ||\phi^1||^2_{H^{-1}(\Omega)}\big) = \\ &= C_0\left\|\{\phi^0,\phi^1\}\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\ . \end{aligned} \tag{7.13}$$

This implies

$$L^2(\Omega)\times H^{-1}(\Omega)\subset F,$$

continuous and densely. To prove that $F \subset L^2(\Omega)\times H^{-1}(\Omega)$. We need to prove the inverse inequality:

$$C_1\left\|\{\phi^0,\phi^1\}\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \le \int_0^T\int_w \phi^2\,dxdt. \tag{7.14}$$

Suppose we have prove (7.14). Then, with (7.13) we conclude the equivalence of the norms $||\{\phi^0,\phi^1\}||_F$ and $\|\{\phi^0,\phi^1\}\|_{L^2(\Omega)\times H^{-1}(\Omega)}$. It follows that we can identify F to $L^2(\Omega)\times H^{-1}(\Omega)$. Consequently its dual is $F' = L^2(\Omega)\times H^1_0(\Omega)$. Therefore given $\{y^1, -y^0\} \in L^2(\Omega)\times H^1_0(\Omega)$ we find a unique $\{\phi^0,\phi^1\} \in L^2(\Omega)\times H^{-1}(\Omega)$, solve the ultra weak boundary value problem (7.3), which gives $\phi = \phi(x,t)$, then the control $h = h(x,t)$ is the restriction of $\phi = \phi(x,t)$ to $w\times]0,T[$. By the regularity of ultra weak solution, proved in Chapter 4, follows that $h \in L^2(w\times]0,T[)$. Consequently everything is in order. ∎

It is important to observe that the inverse inequality (7.14) will be proved for a restricted class of open set w contained in Ω, that is, for w with a particular geometric structure and T large enough.

7.2 The Inverse Inequality

We begin with the notation. As we have done in the Chapter 6, we divide the boundary Γ of Ω in two pieces $\Gamma(x^0)$ and $\Gamma_*(x^0)$, where x^0 is a point of $\mathbb{R}^n$.

We say that $w \subset \Omega$ is a neighborhood, in Ω, of $\overline{\Gamma(x^0)}$, closure of $\Gamma(x^0)$, if there exists some neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of $\overline{\Gamma(x^0)}$ such that

$$w = \Omega \cap \mathcal{O}. \tag{7.15}$$

Observe that $R(x^0)$ was defined in Chapter 6. Then we have the theorem giving the inverse inequality.

Theorem 7.1 *If $T > 2R(x^0)$, exists a constant $C > 0$ such that:*

$$|\phi^0|^2_{L^2(\Omega)} + \|\phi^1\|^2_{H^{-1}(\Omega)} \leq C \int_0^T \int_w \phi^2 \, dxdt \tag{7.16}$$

for all ultra weak solution of (7.3), with $\phi^0 \in L^2(\Omega)$ and $\phi^1 \in H^{-1}(\Omega)$.

Proof: We begin substituting the proof of (7.16) by another equivalent inequality.

Step 1. If exists a constant $C > 0$ such that

$$\|\phi^0\|^2_{H^1_0(\Omega)} + |\phi^1|^2_{L^2(\Omega)} \leq C \int_0^T \int_w \phi'^2 \, dxdt, \tag{7.17}$$

for all weak solution $\phi = \phi(x,t)$ of (7.3) with $\phi^0 \in H^1_0(\Omega)$ and $\phi^1 \in L^2(\Omega)$, then we have the inequality (7.16), for ultra weak solution $\phi = \phi(x,t)$ when we take $\phi^0 \in H^1_0(\Omega) \subset L^2(\Omega)$, $\phi^1 \in L^2(\Omega) \subset H^{-1}(\Omega)$.

In fact, given $\{\phi^0, \phi^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ define $\chi \in H^1_0(\Omega)$ such that $-\Delta\chi = \phi^1$ in Ω. Let us consider,

$$\psi(x,t) = -\chi(x) + \int_0^t \phi(x,s) \, ds,$$

where $\phi = \phi(x,t)$ is the ultra weak solution of (7.3) with initial values ϕ^0, ϕ^1.

If we integrate $(7.3)_1$ we obtain:

$$\phi'(t) - \phi'(0) - \Delta \int_0^t \phi(x,s) \, ds = 0.$$

But $\psi'(x,t) = \phi(x,t)$ and $\psi''(x,t) = \phi'(x,t)$. Then:

$$\psi''(x,t) - \phi^1 + \Delta(\psi(x,t) + \chi(x)) = 0.$$

By the definition of χ, the above equality implies:

$$\left| \begin{array}{l} \psi'' - \Delta\psi = 0 \quad \text{in} \quad Q, \\ \psi = 0 \quad \text{on} \quad \Sigma, \\ \psi(0) = \chi\,, \ \psi'(0) = \phi^0. \end{array} \right. \tag{7.18}$$

Note that in (7.18) we have $\chi \in H^1_0(\Omega)$ and $\phi^0 \in L^2(\Omega)$, what implies the existence of weak solution. If (7.17) is true we have from (7.18),

$$\|\chi\|^2_{H^1_0(\Omega)} + |\phi^0|^2_{L^2(\Omega)} \leq C \int_0^T \int_\Omega \phi^2 \, dxdt \tag{7.19}$$

since $\psi'(x,t) = \phi(x,t)$.

Remark 7.1 *Let us define in $H^{-1}(\Omega)$ an inner product. We know that Δ is an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Let $G = \Delta^{-1}$. Then, for all pair $u, v \in H^{-1}(\Omega)$ we define*

$$(u,v)_{H^{-1}(\Omega)} = \langle u, Gv\rangle_{H^{-1}(\Omega)\times H_0^1(\Omega)} = ((Gu, Gv))_{H_0^1(\Omega)\times H_0^1(\Omega)},$$

which is an inner product in $H^{-1}(\Omega)$. The induced norm is:

$$\|v\|^2_{H^{-1}(\Omega)} = ((Gv, Gv)).$$

Then,

$$\|\phi^1\|^2_{H^{-1}(\Omega)} = ((G\phi^1, G\phi^1)) = ((\chi, \chi)) = \|\chi\|^2_{H_0^1(\Omega)}.$$

By Remark 7.1 we modify (7.19) obtaining:

$$\|\phi^1\|^2_{H^{-1}(\Omega)} + |\phi^0|^2_{L^2(\Omega)} \leq C\int_0^T \int_w \phi^2 \, dxdt.$$

■

Step 2. It follows from the above argument that in order to prove Theorem 7.1 it is sufficient to prove the inequality (7.17) for weak solution $\phi = \phi(x,t)$ of (7.3). We follows Zuazua [70] and Fabre [11] to prove (7.17).

For $T > 2R(x^0)$ we know by Chapter 6, (6.22), that:

$$\int_\Omega (|\nabla\phi^0(x)|^2 + |\phi^1(x)|^2)\, dx \leq \frac{R(x^0)}{T - 2R(x^0)} \int_{\Sigma(x^0)} \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt$$

for the weak solution $\phi = \phi(x,t)$ of (7.3).

Now, for $\varepsilon > 0$, $T - 2\varepsilon > 2R(x^0)$ we have:

$$E(0) = \frac{1}{2}\int_\Omega (|\nabla\phi^0(x)|^2 + |\phi^1(x)|^2)\, dx \leq C\int_\varepsilon^{T-\varepsilon} \int_{\Gamma(x^0)} \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \tag{7.20}$$

after the change of variables $\tau = (T - 2\varepsilon)t + T\varepsilon$, $0 \leq t \leq T$ that implies $\varepsilon \leq \tau \leq T - \varepsilon$.

Consider $h \in [C^1(\overline{\Omega})]^n$ such that $h \cdot \nu \geq 0$ for all $x \in \Gamma$, $h = \nu$ on $\Gamma(x^0)$ and $h = 0$ on $\Omega\backslash w$. Let be $\eta \in C^1([0,T])$ such that $\eta(0) = \eta(T) = 0$, $\eta(t) = 1$ in $]\varepsilon, T-\varepsilon[$. We define $q(x,t) = \eta(t)h(x)$ which belongs to $W^{1,\infty}(\Omega)$ and satisfies:

$$\left|\begin{array}{l} \text{(i)} \quad q(x,t) = \nu(x) \text{ for all } (x,t) \in \Gamma(x^0)\times]\varepsilon, T-\varepsilon[\,; \\ \text{(ii)} \quad q(x,t)\cdot\nu(x) \geq 0 \text{ for all } (x,t) \in \Gamma\times]0,T[\,; \\ \text{(iii)} \quad q(x,0) = q(x,T) = 0 \text{ for all } x \in \Omega; \\ \text{(iv)} \quad q(x,t) = 0 \text{ for all } (x,t) \in (\Omega\backslash w)\times]0,T[\,. \end{array}\right. \tag{7.21}$$

If we consider the multiplier $q_k \dfrac{\partial \phi}{\partial x_k}$ we obtain the following identity for all weak solution $\phi = \phi(x,t)$ of (7.3):

$$\begin{aligned}\frac{1}{2}\int_{\Sigma} q\cdot\nu\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt &= (\phi'(t), q\cdot\nabla\phi)\Big|_0^T + \\ &+\frac{1}{2}\int_Q \operatorname{div} q(|\phi'|^2 - |\nabla\phi|^2)\,dxdt + \\ &+\int_Q \frac{\partial q_k}{\partial x_j}\frac{\partial\phi}{\partial x_k}\frac{\partial\phi}{\partial x_j}\,dxdt - \int_Q \phi' q'\cdot\nabla\phi\;dxdt.\end{aligned}$$

Applying this identity with the above defined vector field q, we obtain:

$$\frac{1}{2}\int_0^T\int_\Gamma q\cdot\nu\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \geq \frac{1}{2}\int_\varepsilon^{T-\varepsilon}\int_{\Gamma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt$$

because $q(x,t)=\nu$ on $\Gamma(x^0)\times]\varepsilon, T-\varepsilon[$, and

$$(\phi'(t), q\cdot\nabla\phi)\Big|_0^T = 0, \quad \text{because } \eta(0)=\eta(T)=0.$$

Since $q\in C^1(\overline{\Omega}\times]0,T[)$, div q is bounded. We also have:

$$\begin{aligned}\left|\int_Q \frac{\partial q_k}{\partial x_j}\frac{\partial\phi}{\partial x_k}\frac{\partial\phi}{\partial x_j}\,dxdt\right| &\leq C\sum_{k,j}\int_{w\times]0,T[}\frac{\partial\phi}{\partial x_k}\frac{\partial\phi}{\partial x_j}\,dxdt \overset{\text{Cauchy inequality}}{\leq} \\ &\leq C\int_0^T\int_w |\nabla\phi|^2\,dxdt \leq C\int_0^T\int_w (|\phi'|^2 + |\nabla\phi|^2)\,dxdt.\end{aligned}$$

The same argument to estimate $-\displaystyle\int_Q \phi' q'\cdot\nabla\phi\,dxdt$. Then,

$$\int_\varepsilon^{T-\varepsilon}\int_{\Gamma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \leq C\int_0^T\int_w (|\phi'|^2+|\nabla\phi|^2)\,dxdt, \tag{7.22}$$

where $C>0$ is a constant depending on $\|q\|_{W^{1,\infty}(\Omega)}$.

From (7.20) and (7.22) we get:

$$E(0) \leq C\int_\varepsilon^{T-\varepsilon}\int_w (|\phi'|^2+|\nabla\phi|^2)\,dxdt, \tag{7.23}$$

Step 3. We prove in this step that:

$$\left\|\phi^0\right\|_{H_0^1(\Omega)} + \left|\phi^1\right|^2_{L^2(\Omega)} \leq C\int_0^T\int_w \phi'^2\,dxdt + C\int_0^T\int_w \phi^2\,dxdt. \tag{7.24}$$

In fact, let $w_0\subset\Omega$ be a neighborhood of $\overline{\Gamma(x^0)}$ such that:

$$\Omega\cap w_0 \subset w.$$

Note that (7.23) is true for each neighborhood of $\overline{\Gamma(x^0)}$, then it is correct for w_0. We obtain:

$$E(0) \leq C \int_{\varepsilon}^{T-\varepsilon} \int_{w_0} (\phi'^2 + |\nabla\phi|^2)\, dxdt. \tag{7.25}$$

Consider $\rho \in W^{1,\infty}(\Omega)$, $\rho \geq 0$ such that

$$\rho(x) = 1 \quad \text{in} \quad w_0 \quad \text{and} \quad \rho(x) = 0 \quad \text{in} \quad \Omega \backslash w.$$

Define $p(x,t)$ in Q by

$$p(x,t) = \eta(t)\rho^2(x)$$

where $\eta(t)$ is the function above defined. We have:

$$\left| \begin{array}{ll} \text{(i)} & p(x,t) = 1 \text{ in } w_0 \times]\varepsilon, T-\varepsilon[\,; \\ \text{(ii)} & p(x,t) = 0 \text{ in } (\Omega\backslash w)\times]0,T[\,; \\ \text{(iii)} & p(x,0) = p(x,T) = 0 \text{ in } \Omega; \\ \text{(iv)} & \dfrac{|\nabla p|}{p} \in L^\infty(Q). \end{array} \right. \tag{7.26}$$

Multiply both sides of $(7.3)_1$ by $p\phi$ and integrate by parts in Q. We obtain:

$$\int_Q p\phi\phi''\, dxdt - \int_Q p\phi\Delta\phi\, dxdt = 0. \tag{7.27}$$

Analysis of the first integral

$$\int_0^T (\phi'', p\phi)\, dt = (\phi', p\phi)\Big|_0^T - \int_0^T (\phi', p\phi')\, dt - \int_0^T (p'\phi, \phi')\, dt,$$

$p(x,0) = p(x,T) = 0$, then:

$$\int_0^T (\phi'', p\phi)\, dt = -\int_0^T (\phi', p\phi')\, dt - \int_0^T (p'\phi, \phi')\, dt. \tag{7.28}$$

Analysis of the second integral

$$-\int_\Omega \Delta\phi \cdot p\phi\, dxdt = \int_\Omega \nabla\phi \cdot \nabla(p\phi)\, dx - \int_\Gamma p\phi \frac{\partial\phi}{\partial\nu}\, d\Gamma.$$

The surface integral on Γ is zero because ϕ is solution of (7.3). Then:

$$-\int_\Omega \Delta\phi \cdot p\phi\, dxdt = \int_0^T\int_w p\nabla\phi \cdot \nabla\phi\, dxt + \int_0^T\int_w \nabla p \cdot \nabla\phi\phi\, dxdt, \tag{7.29}$$

because $p(x,t) = 0$ in $\Omega\backslash w$, $p(x,t) = \eta(t)\rho^2(x)$.

We then obtain from (7.27), (7.28) and (7.29):

$$\begin{aligned}\int_0^T\int_w p|\nabla\phi|^2\,dxdt = \int_0^T\int_w p\phi'^2\,dxdt + \int_0^T\int_w p'\phi\phi'\,dxdt - \\ - \int_0^T\int_w \nabla p\cdot\nabla\phi\phi\;dxdt.\end{aligned} \tag{7.30}$$

By (7.30) and (7.26) we obtain:

$$\begin{aligned}\int_0^T\int_w p|\nabla\phi|^2\,dxdt \le C\int_0^T\int_w (|\phi'|^2+|\phi|^2)\,dxdt + \\ + \left|\int_0^T\int_w \nabla p\cdot\nabla\phi\phi\,dxdt\right|.\end{aligned} \tag{7.31}$$

By (7.30) we obtain:

$$\begin{aligned}\left|\int_0^T\int_w \nabla p\cdot\nabla\phi\phi\;dxdt\right| \le \frac{1}{2}\int_0^T\int_w p|\nabla\phi|^2\,dxdt + \\ + \frac{1}{2}\int_0^T\int_w \frac{|\nabla p|^2}{p}\,\phi^2\,dxdt.\end{aligned} \tag{7.32}$$

By (7.31), (7.32) and (7.26) we have:

$$\int_\varepsilon^{T-\varepsilon}\int_w |\nabla\phi|^2\;dxdt \le C\int_0^T\int_w (\phi'^2+\phi^2)\,dxdt. \tag{7.33}$$

By (7.33) and (7.25) follows:

$$\left\|\phi^0\right\|^2_{H_0^1(\Omega)} + |\phi'|^2_{L^2(\Omega)} \le C\int_0^T\int_w \phi'^2\;dxdt + C\int_0^T\int_w \phi^2\;dxdt. \tag{7.34}$$

■

From (7.34) and hidden regularity, Chapter 4, we obtain

$$\int_{\Sigma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \le C\int_0^T\int_w (\phi'^2+\phi^2)\,dxdt. \tag{7.35}$$

Step 5. Suppose (7.17) is not true. Then given a natural number n exists initial data $\widetilde{\phi}_n^0$, $\widetilde{\phi}'_n$ such that the solution $\widetilde{\phi}_n$ of (7.3) corresponding to this initial conditions satisfies:

$$\left\|\widetilde{\phi}_n^0\right\|^2_{H_0^1(\Omega)} + \left|\widetilde{\phi}_n^1\right|^2_{L^2(\Omega)} \ge n\left\|\widetilde{\phi}'_n\right\|^2_{L^2(0,T;L^2(w))}\;.$$

Let us define

$$K = \left(\left\|\widetilde{\phi}_n^0\right\|^2_{H_0^1(\Omega)} + \left|\widetilde{\phi}_n^1\right|^2_{L^2(\Omega)}\right)^{1/2},$$

and

$$\phi_n^0 = \frac{\widetilde{\phi}_n^0}{K}\,; \quad \phi_n^1 = \frac{\widetilde{\phi}_n^1}{K}\,; \quad \phi_n = \frac{\widetilde{\phi}_n}{K}\,.$$

We obtain

$$\left| \begin{array}{l} \|\phi_n'\|^2_{L^2(0,T;L^2(w))} \leq \dfrac{1}{n} \\ \|\phi_n^0\|_{H_0^1(\Omega)} + |\phi_n^1|^2_{L^2(\Omega)} = 1 \end{array} \right. \tag{7.36}$$

From (7.36) we obtain:

$$\varliminf_{n\to\infty} \int_0^T \int_w \phi_n'^2 \, dxdt = 0. \tag{7.37}$$

From (7.36) we also obtain subsequences such that:

$$\phi_n^0 \rightharpoonup \phi^0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{and} \quad \phi_n^1 \rightharpoonup \phi^1 \quad \text{in} \quad L^2(\Omega)$$

weakly. The solution ϕ_n of (7.3) corresponding to the initial data ϕ_n^0 , ϕ_n^1 has the estimates:

$$\left| \begin{array}{ll} \phi_n & \text{is bounded in} \quad L^\infty(0,T;H_0^1(\Omega)), \\ \phi_n' & \text{is bounded in} \quad L^\infty(0,T;L^2(\Omega)). \end{array} \right. \tag{7.38}$$

The estimate (7.38) is true in w instead of Ω. Then, there exists a subsequence ϕ_n such that

$$\left| \begin{array}{ll} \phi_n \rightharpoonup \phi & \text{weak star in} \quad L^\infty(0,T;H_0^1(\Omega)), \\ \phi_n' \rightharpoonup \phi' & \text{weak star in} \quad L^\infty(0,T;L^2(\Omega)). \end{array} \right. \tag{7.39}$$

Since $H_0^1(\Omega) \subset L^2(\Omega)$ is compact, the estimates (7.38) and Aubin-Lions compactness theorem, we obtain a subsequence ϕ_n such that

$$\phi_n \to \phi \quad \text{strongly in} \quad L^2(0,T;L^2(w)). \tag{7.40}$$

From (7.37), $(7.39)_2$ and Banach-Steinhauss theorem, it follows that $\phi'(x,t) = 0$ on $w\times]0,T[$, that is, $\phi(x,t)$ is constant with respect to t in $w\times]0,T[$. But $\phi = 0$ on Σ_0 because ϕ is solution of (7.3). Then $\phi(x,t) = 0$ on $w\times]0,T[$, by Holmgren's theorem. Then, by (7.40) we obtain:

$$\phi_n \to 0 \quad \text{strongly in} \quad L^2(0,T;L^2(w)).$$

Then by (7.35) for ϕ_n we obtain:

$$\frac{\partial \phi_n}{\partial \nu} \to 0 \quad \text{in} \quad L^2(\Sigma_0).$$

By hidden inequality it follows that $\phi_n^0 \to 0$ in $H_0^1(\Omega)$ and $\phi_n^1 \to 0$ in $L^2(\Omega)$ what is a contradiction with $(7.36)_2$. ∎

Chapter 8

Exact Controllability for Timoshenko System

8.1 Exact Controllability for Timoshenko System

In this section we are interested in the exact controllability of the system:

$$\left| \begin{array}{l} y'' - ay_{xx} - z_x + y = 0 \\ z'' - bz_{xx} + y_x = 0 \end{array} \right. \tag{8.1}$$

which is motivated by questions of one dimensional elasticity. In fact, the system (8.1) has its origin in the study of transverse vibrations of beams when we consider the efects of rotatory inertia. It is called, by S. Timoshenko [68], model for transverse vibrations of a beam when we consider rotatory inertia and shearing deformation. Note that a and b are positive constants. We suppose the beam of lenght $L = 1$. The transverse displacement of the point x, for $0 \leq x \leq 1$ at the instant t, $0 \leq t \leq T$, that is, the deformation curve, is represented by $z = z(x,t)$. We denote by $y = y(x,t)$ the slope of the deformation curve $z = z(x,t)$ motivated by the rotatory action.

Let us represent by Ω the segment $[0,1]$ of the real line $\mathbb{R}$, which represents the beam in equilibrium. By Q we represent the rectangle $\Omega \times]0,T[$ of the plane $\mathbb{R}^2$, where T is a positive real number. We denote by y' and y_x the derivatives, respectively, with respect to t and x, of the function $y = y(x,t)$. We then consider the non homogeneous mixed problem:

$$\left| \begin{array}{l} y'' - ay_{xx} - z_x + y = 0 \quad \text{in} \quad Q, \\ z'' - bz_{xx} + y_x = 0 \quad \text{in} \quad Q, \end{array} \right. \tag{8.2}$$

$$\left| \begin{array}{ll} y(0,T) = v(t), \ y(1,t) = 0 & \text{in} \quad]0,T[\,, \\ z(0,T) = w(t), \ z(1,t) = 0 & \text{in} \quad]0,T[\,. \end{array} \right. \tag{8.3}$$

$$\left| \begin{array}{ll} y(x,0) = y^0(x), \ y'(x,0) = y^1 & \text{in} \quad \Omega\,, \\ z(x,0) = z^0(x), \ z'(x,0) = z^1 & \text{in} \quad \Omega\,. \end{array} \right. \tag{8.4}$$

The exact controllability for (8.1) is formulated as follows: given $T > 0$ find a Hilbert space H such that for every set $\{y^0,\ y^1\}$, $\{z^0, z^1\}$ in H, there exists a pair of controls $v(t)$, $w(t)$ in $L^2(0,T)$ such that the solution $y = y(x,t)$, $z = z(x,t)$ of (8.2), (8.3) and (8.4) satisfy the condition:

$$\left| \begin{array}{ll} y(x,T) = 0, \ y'(x,T) = 0 & \text{in} \quad \Omega, \\ z(x,T) = 0, \ z'(x,T) = 0 & \text{in} \quad \Omega. \end{array} \right. \tag{8.5}$$

In the present section we solve the above problem of exact controllability for the system (8.1) by HUM, idealized by J.L. Lions [36] and [38]. For the case when a, b are variable cf. Medeiros [47].

Plan of this Chapter.

- Controllability of the Timoshenko system by HUM.
- Basic results on solutions of the Timoshenko system.
- Energy Inequalities.
- Direct and Inverse Inequalities.
- Non homogeneous mixed problem for the Timoshenko system. Ultra weak solutions.

8.2 Exact Controllability for the Timoshenko System by HUM

In the present section it is described how to apply HUM in the present situation. A summary of proofs of the properties of solution is done in the next paragraphs.

Theorem 8.1 *Suppose a and b real numbers such that:*

$$\min\{a,b\} > 1 \quad and \quad \alpha = \max\left\{\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}\right\},$$

and let be $T > 2\alpha$. Then, for each set of initial values $\{y^0, y^1\}$ and $\{z^0, z^1\}$ in $L^2(0,1) \times H^{-1}(0,1)$ exists a pair of controls $v(t)$ and $w(t)$ in $L^2(0,T)$ such that the solution $y = y(x,t)$ and $z = z(x,t)$ of (8.2), (8.3) and (8.4) satisfies the condition (8.5).

Proof: The proof by HUM will be done in the following steps.

Step 1. Given $\{\phi^0, \phi^1\}$ and $\{\psi^0, \psi^1\}$ in $\mathcal{D}(0,1)\times\mathcal{D}(0,1)$ we solve the regular homogeneous mixed problem:

$$\left| \begin{array}{l} \phi'' - a\phi_{xx} - \psi_{xx} + \phi = 0 \quad \text{in} \quad Q, \\ \psi'' - b\psi_{xx} + \phi_x = 0 \quad \text{in} \quad Q, \end{array} \right. \tag{8.6}$$

$$\left| \begin{array}{l} \phi(0,t) = 0,\ \phi(1,t) = 0 \quad \text{in} \quad \Omega, \\ \psi(0,t) = 0,\ \psi(1,t) = 0 \quad \text{in} \quad \Omega, \end{array} \right. \tag{8.7}$$

$$\left| \begin{array}{l} \phi(x,0) = \phi^0(x),\ \phi'(x,0) = \phi^1(x) \quad \text{in} \quad \Omega, \\ \psi(x,0) = \psi^0(x),\ \psi'(x,0) = \psi^1(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{8.8}$$

The above mixed problem (8.6), (8.7) and (8.8) has only one solution $\phi = \phi(x,t)$, $\psi = \psi(x,t)$ satisfying:

$$\phi_x(0,t), \psi_x(0,t) \quad \text{are in} \quad L^2(0,T). \tag{8.9}$$

Step 2. With the solution $\phi = \phi(x,t)$ and $\psi = \psi(x,t)$ of Step 1, we solve the backward problem:

$$\left| \begin{array}{l} \xi'' - a\xi_{xx} - \zeta_x + \xi = 0 \quad \text{in} \quad Q, \\ \zeta'' - b\zeta_{xx} + \xi_x = 0 \quad \text{in} \quad Q. \end{array} \right. \tag{8.10}$$

$$\left| \begin{array}{l} \xi(0,t) = -a\phi_x(0,t),\ \xi(1,t) = 0 \quad \text{in} \quad]0,T[\,, \\ \zeta(0,t) = -b\psi_x(0,t),\ \zeta(1,t) = 0 \quad \text{in} \quad]0,T[\,, \end{array} \right. \tag{8.11}$$

$$\left| \begin{aligned} &\xi(x,T)=0,\ \xi'(x,T)=0 \quad \text{in} \quad \Omega, \\ &\zeta(x,T)=0,\ \zeta'(x,T)=0 \quad \text{in} \quad \Omega. \end{aligned} \right. \tag{8.12}$$

Note that (8.10), (8.11) and (8.12) has only one solution $\xi = \xi(x,t)$ and $\zeta = \zeta(x,t)$.

The operator Λ. For all $\{\phi^0, \phi^1\}$, $\{\psi^0, \psi^1\}$ in $\mathcal{D}(0,1) \times \mathcal{D}(0,1)$ we solve (8.6), (8.7) and (8.8). With the solution $\phi = \phi(x,t)$, $\psi = \psi(x,t)$, satisfying (8.9), we solve (8.10), (8.11) and (8.12), obtaining $\xi = \xi(x,t)$, $\zeta = \zeta(x,t)$, making sense to calculate $\xi(0) = \xi(x,0)$, $\zeta(0) = \zeta(x,0)$, $\xi'(0) = \xi'(x,0)$ and $\zeta'(0) = \zeta'(x,0)$. Then, is well defined the map:

$$\Lambda\{\phi^0, \phi^1, \psi^0, \psi^1\} = \{\xi'(0), -\xi(0), \zeta'(0), -\zeta(0)\} \tag{8.13}$$

for all $\{\phi^0, \phi^1\}$, $\{\psi^0, \psi^1\}$ in $\mathcal{D}(0,1) \times \mathcal{D}(0,1)$.

Step 3. Multiply both sides of $(8.6)_1$ by ξ and $(8.6)_2$ by ζ, solution of (8.10), (8.11) and (8.12), and integrate in Q. We obtain, after integration by parts:

$$\begin{aligned} &(\xi'(0), \phi^0) - (\xi(0), \phi^1) + (\zeta'(0), \psi^0) - (\zeta(0), \psi^1) = \\ &= \int_0^T a\phi_x^2(0,t)\,dt + \int_0^T b\psi_x^2(0,t)\,dt. \end{aligned} \tag{8.14}$$

Observe that the second member of (8.14) is a consequence of the boundary conditions $\xi(0,t) = -a\phi_x(0,t)$ and $\zeta(0,t) = -b\psi_x(0,t)$.

From (8.13) and (8.14) we obtain:

$$\begin{aligned} &\langle \Lambda\{\phi^0, \phi^1, \psi^0, \psi^1\}, \{\phi^0, \phi^1, \psi^0, \psi^1\} \rangle = \\ &= (\{\xi'(0), -\xi(0), \zeta'(0), -\zeta(0)\}, \{\phi^0, \phi^1, \psi^0, \psi^1\})_F = \\ &= (\xi'(0), \phi^0) - (\xi(0), \phi^1) + (\zeta'(0), \psi^0) - (\zeta(0), \psi^1) = \\ &= \int_0^T a\phi_x^2(0,t)\,dt + \int_0^T b\psi_x^2(0,t)\,dt. \end{aligned} \tag{8.15}$$

We shall prove at the end of the section, cf. §4, the existence of positive constants C_0, C_1 such that:

$$\begin{aligned} &C_0 \left\|\{\phi^0, \phi^1, \psi^0, \psi^1\}\right\|_F^2 \\ &\leq \int_0^T a\phi_x^2(0,t)\,dt + \int_0^T b\psi_x^2(0,t)\,dt \leq \\ &\leq C_1 \left\|\{\phi^0, \phi^1, \psi^0, \psi^1\}\right\|_F^2 . \end{aligned} \tag{8.16}$$

Note that in (8.16) we have, by definition

$$\left\|\{\phi^0, \phi^1, \psi^0, \psi^1\}\right\|_F^2 = \int_0^1 (|\phi^1(x)|^2 + |\phi_x^0(x)|^2 + |\psi^1(x)|^2 + |\psi_x^0(x)|^2)\,dx, \tag{8.17}$$

which is a norm in $(H_0^1(0,1) \times L^2(0,1))^2$.

By (8.16) it follows that $\|\{\phi^0, \phi^1, \psi^0, \psi^1\}\|_F$ defined by (8.17) is a norm in $(\mathcal{D}(0,1) \times \mathcal{D}(0,1))^2$, equivalent to the norm of $(H_0^1(0,1) \times L^2(0,1))^2$ defined by (8.17). The operator Λ defined by (8.13) is linear and continuous with respect to the norm $|| \cdot ||_F$. Then it has a unique extension, by continuity, to the closure of $(\mathcal{D}(0,1) \times \mathcal{D}(0,1))^2$ with respect to $|| \ ||_F$, which, by (8.16) is equivalent to the norm of $(H_0^1(0,1) \times L^2(0,1))^2$ given by (8.17). Therefore, $F = (H_0^1(0,1) \times L^2(0,1))^2$ and we have

$$\Lambda\colon F \to F'; \tag{8.18}$$

F' dual of F, because Λ is also coercive. This (8.18) is a consequence of Lax-Milgram lemma. Note, also, that $F' = (H^{-1}(0,1) \times L^2(0,1))^2$.

It then follows that Λ is an isomorphism between $F = (H_0^1(0,1) \times L^2(0,1))^2$ and its dual $F' = (H^{-1}(0,1) \times L^2(0,1))^2$. Consequently, given $\{y^0, y^1, z^0, z^1\}$ such that $\{y^1, -y^0\}, \{z^1, -z^0\} \in H^{-1}(0,1) \times L^2(0,1)$, the equation

$$\Lambda\{\phi^0, \phi^1, \psi^0, \psi^1\} = \{y^1, -y^0, z^1, -z^0\}, \tag{8.19}$$

has a unique solution $\{\phi^0, \phi^1, \psi^0, \psi^1\}$ such that $\{\phi^0, \phi^1\}, \{\psi^0, \psi^1\} \in H_0^1(0,1) \times L^2(0,1)$.

By (8.13) and (8.19) we conclude that the unique solution cf (8.10), (8.11) and (8.12) satisfies (8.4). Then, the unique solution of (8.2), (8.3) and (8.4) with controls:

$$v(t) = -a\phi_x(0,t) \quad \text{and} \quad w(t) = -b\psi_x(0,t) \tag{8.20}$$

satisfies (8.5), what we would like to prove. ■

8.3 Basic Results on Solutions of the Timoshenko System

Let us begin with the study of regularity for solutions of the following mixed problem:

$$\left| \begin{array}{l} \phi'' - a\phi_{xx} - \psi_x + \phi = f \quad \text{in} \quad Q, \\ \psi'' - b\psi_{xx} + \phi_x = g \quad \text{in} \quad Q, \end{array} \right. \tag{8.21}$$

$$\left| \begin{array}{l} \phi(0,t) = 0, \ \phi(1,t) = 0 \quad \text{in} \quad]0,T[\,, \\ \psi(0,t) = 0, \ \psi(1,t) = 0 \quad \text{in} \quad]0,T[\,, \end{array} \right. \tag{8.22}$$

$$\left| \begin{array}{l} \phi(x,0) = \phi^0(x), \ \phi'(x,0) = \phi^1(x) \quad \text{in} \quad \Omega, \\ \psi(x,0) = \psi^0(x), \ \psi'(x,0) = \psi^1(x) \quad \text{in} \quad \Omega. \end{array} \right. \tag{8.23}$$

Theorem 8.2 *Given*

$$\phi^0, \psi^0 \in H_0^1(0,1); \ \phi^1, \psi^1 \in L^2(0,1); \ f, g \in L^1(0,1; L^2(0,1)), \tag{8.24}$$

exists only one weak solution $\{\phi, \psi\}$ *of (8.21), (8.22) and (8.23) satisfying the conditions:*

$$\phi, \psi \in L^\infty(0,T; H_0^1(0,1)), \tag{8.25}$$

$$\phi', \psi' \in L^\infty(0,T; L^2(0,1)). \tag{8.26}$$

The mapping

$$\{\{\phi^0, \psi^0\}, \{\phi^1, \psi^1\}, \{f, g\}\} \to \{\{\phi, \psi\}, \{\phi', \psi'\}\} \tag{8.27}$$

is continuous.

Theorem 8.3 *Given*

$$\begin{array}{c} \phi^0, \psi^0 \in H_0^1(0,1) \cap H^2(0,1); \phi^1, \psi^1 \in H_0^1(0,1); \\ f, g \in W^{1,1}(0,T; H_0^1(0,1)) \end{array} \tag{8.28}$$

exists only one strong solution of (8.21), (8.22) and (8.23) satisfying the conditions:

$$\phi, \psi \in L^\infty(0,T; H_0^1(0,1) \cap H^2(0,1)), \tag{8.29}$$

$$\phi', \psi' \in L^\infty(0,T; H_0^1(0,1)). \tag{8.30}$$

The mapping

$$\{\{\phi^0, \psi^0\}, \{\phi^1, \psi^1\}, \{f, g\}\} \to \{\{\phi, \psi\}, \{\phi', \psi'\}\}$$

is continuous.

Proof of the Theorem 8.2.

By Galerkin method we prove existence of local solution. The a priori estimates permits to extend the solution and obtain, in the limit, the unique solution. It is shown how to obtain these estimates.

In fact, multiply both sides of $(8.21)_1$ by ϕ' and of $(8.21)_2$ by ψ' and integrate on $(0,1)$. We obtain:

$$\left| \begin{array}{l} (\phi'', \phi') - a(\phi_{xx}, \phi') - (\psi_x, \phi') + (\phi, \phi') = (f, \phi') \\ (\psi'', \psi') - b(\psi_{xx}, \psi') + (\phi_x, \psi') = (g, \psi') \end{array} \right. \tag{8.31}$$

Note that $(\cdot,\cdot)$ is the inner product in $L^2(0,1)$. Then, from (8.31) we obtain:

$$\left| \begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(|\phi'(t)|^2 + a||\phi(t)||^2 + |\phi(t)|^2\right) = (f, \phi') + (\psi_x, \phi') \\ \frac{1}{2}\frac{d}{dt}\left(|\psi'(t)|^2 + b||\psi(t)||^2\right) = (g, \psi') - (\phi_x, \psi') \end{array} \right. \tag{8.32}$$

Observe that $|\cdot|$, $||\cdot||$ are the norm, respectively, in $L^2(0,1)$ and $H^1(0,1)$. By addition of the equations (8.32) we obtain:

$$\begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(|\phi'(t)|^2 + |\psi'(t)|^2 + a||\phi(t)||^2 + b||\psi(t)||^2 + |\phi(t)|^2\right) = \\ = (f, \phi') + (g, \psi') + (\psi_x, \phi') - (\phi_x, \psi'). \end{array} \tag{8.33}$$

Let us define the energy associated to (8.21), (8.22) and (8.23) by:

$$E(t) = \frac{1}{2}\left(|\phi'(t)|^2 + |\psi'(t)|^2 + a||\phi(t)||^2 + b||\psi(t)||^2 + |\phi(t)|^2\right) \tag{8.34}$$

From (8.33) and (8.32) it follows:

$$\begin{aligned} E'(t) \le \frac{1}{2}\big(&|f(t)| + |f(t)|\,|\phi'(t)|^2 + |g(t)| + |g(t)|\,|\psi'(t)|^2 + \\ &+ ||\psi(t)||^2 + |\phi'(t)|^2 + ||\phi(t)||^2 + |\psi'(t)|^2\big). \end{aligned}$$

Then:

$$E'(t) \le \frac{1}{2}\big(|f(t)| + |g(t)|\big) + h(t)E(t), \tag{8.35}$$

where

$$h(t) = |f(t)| + |g(t)| + \frac{1}{a} + \frac{1}{b} \in L^2(0,T).$$

Integrating (8.35) we obtain:

$$E'(t) \le E(0) + \frac{1}{2}\int_0^T \big(|f(t)| + |g(t)|\big)\,dt + \int_0^t h(s)E(s)\,ds. \tag{8.36}$$

From Gronwall's inequality we have from (8.36):

$$E(t) \le C\big(\,\|f\|_{L^1(0,T;L^2(0,1))} + \|g\|_{L^1(0,T;L^2(\Omega))} + E_0\big) \tag{8.37}$$

where $E_0 = E(0)$.

From (8.37) it follows that Galerkin approximations are bounded in $L^\infty(0,T;L^2(0,1))$ and $L^\infty(0,T;H_0^1(0,1))$, which is sufficient to obtain what claims Theorem 8.2. ■

Proof of the Theorem 8.3.

The same remark done in the proof of the Theorem 8.2, about Galerkin approximations, is true here. Then we will do a priori estimates.

In fact, multiply both sides of $(8.21)_1$ by $-\phi'_{xx}$ and of $(8.21)_2$ by $-\psi'_{xx}$ and integrate on $]0,1[$. We obtain:

$$\left| \begin{array}{l} (\phi'', -\phi'_{xx}) - a(\phi_{xx}, -\phi'_{xx}) - (\psi_x, \phi'_{xx}) + (\phi, -\phi'_{xx}) = (f, -\phi'_{xx}) \\ (\psi'', -\psi'_{xx}) - b(\psi_{xx}, -\psi'_{xx}) + (\phi_x, -\psi'_{xx}) = (g, -\psi'_{xx}) \end{array} \right. \tag{8.38}$$

Integrating by parts (8.38), we have:

$$\left| \begin{array}{l} \frac{1}{2}\frac{d}{dt}\left(|\phi'_x|^2 + a|\phi_{xx}|^2 + |\phi_x|^2\right) = (f_x, \phi'_x) - (\psi_x, \phi'_{xx}) \\ \frac{1}{2}\frac{d}{dt}\left(|\psi'_x|^2 + b|\psi_{xx}|^2\right) = (g_x, \psi'_x) + (\phi_x, \psi'_{xx}) \end{array} \right. \tag{8.39}$$

Remark 8.1 *Note that*

$$\frac{d}{dt}(\psi_x, \phi_{xx}) = (\psi'_x, \phi_{xx}) + (\psi_x, \phi'_{xx}).$$

Integrating from 0 to t, we obtain:

$$\int_0^t (\psi_x, \phi'_{xx})\,dt = (\psi_x(t), \phi_{xx}(t)) - (\psi_x^0, \phi_{xx}^0) - \int_0^t (\psi'_x, \phi_{xx})\,ds.$$

■

Integrating $(8.39)_1$ from 0 to t and observing Remark 8.1, we have from $(8.39)_1$

$$\begin{array}{l} \frac{1}{2}\left(|\phi'_x|^2 + a|\phi_{xx}|^2 + |\phi_x|^2\right) = \frac{1}{2}\left(|\phi_x^1|^2 + a|\phi_{xx}^0|^2 + |\phi_x^0|^2\right) + \\ + \int_0^t (f_x, \phi'_x)\,ds - (\psi_x(t), \phi_{xx}(t)) + (\psi_x^0, \phi_{xx}^0) + \int_0^t (\psi'_x, \phi_{xx})\,ds. \end{array} \tag{8.40}$$

By similar argument, from $(8.39)_2$ we get:

$$\begin{array}{l} \frac{1}{2}\left(|\psi'_x|^2 + b|\psi_x|^2\right) = \frac{1}{2}\left(|\psi_x^1|^2 + b|\psi_x^0|^2\right) + \\ + \int_0^T (g_x, \psi'_x)\,ds + (\phi_x(t), \psi_{xx}(t)) - (\phi_x^0, \psi_{xx}^0) - \int_0^t (\phi'_x, \psi_{xx})\,ds. \end{array} \tag{8.41}$$

Adding (8.40) and (8.41) we get:

$$\begin{aligned}
&\frac{1}{2}\left[|\phi_x'|^2 + a|\phi_{xx}|^2 + |\phi_x|^2 + |\psi_x'|^2 + b|\psi_x|^2\right] \leq \\
&\leq \frac{1}{2}\left(|\phi_x^1|^2 + a|\phi_{xx}^0|^2 + |\phi_x^0|^2 + |\psi_x^1|^2 + |\psi_x^0|^2\right) + \\
&+ \int_0^t |f_x|\,|\phi_x'|\,ds + \int_0^t |g_x|\,|\psi_x'|\,ds + |\psi_x|\,|\phi_{xx}| + \\
&+|\psi_x^0|\,|\phi_{xx}^0| + |\phi_x|\,|\psi_{xx}| + |\phi_x^0|\,|\psi_{xx}^0|.
\end{aligned} \tag{8.42}$$

We define:

$$F(t) = \frac{1}{2}\left(|\phi_x'(t)|^2 + a|\phi_{xx}(t)|^2 + |\psi_x'(t)|^2 + b|\psi_x(t)|^2 + |\phi_x(t)|^2\right),$$

then we obtain from (8.42):

$$F(t) \leq C\left(\|f\|_{L^1(0,T;H_0^1(0,1))} + \|g\|_{L^1(0,T;H_0^1(0,1))} + F(0)\right).$$

From this estimate we obtain the proof of the Theorem 8.3. ■

8.4 Energy inequalities

In this section we will prove that the energy defined by (8.34) associated to the systems (8.6), (8.7) and (8.8), satisfies an inequality of the type:

$$C_0 E_0 \leq E(t) \leq C_1 E_0 \quad \text{for all} \quad 0 \leq t \leq T, \tag{8.43}$$

where C_0 , C_1 are positive constants.

In fact, multiply $(8.6)_1$ by ϕ' and $(8.6)_2$ by ψ' and integrate on $]0,1[$. Whence,

$$\left| \begin{aligned} &(\phi'', \phi') - a(\phi_{xx}, \phi') + (\psi, \phi_x) + (\phi, \phi') = 0 \\ &(\psi'', \psi') - b(\psi_{xx}, \psi') + (\phi_x, \psi') = 0 \end{aligned} \right. \tag{8.44}$$

From (8.44) it follows:

$$E'(t) + \frac{d}{dt}(\psi, \phi_x) = 0$$

and integrating we have:

$$E(t) + (\psi(t), \phi_x(t)) = E_0 + (\psi^0, \phi_x^0).$$

We know that:

$$-\frac{1}{2}\left(|\psi^0|^2 + |\phi_x^0|^2\right) \leq (\psi^0, \phi_x^0),$$

therefore,

$$E_0 - \frac{1}{2}\left(|\psi^0|^2 + |\phi_x^0|^2\right) \leq E_0 + (\psi^0, \phi_x^0). \tag{8.45}$$

We have:

$$\begin{aligned} E_0 - \frac{1}{2}\left(|\psi^0|^2 + |\phi_x^0|^2\right) \geq \frac{1}{2}|\phi^1|^2 + \frac{1}{2}|\psi^1|^2 + \frac{1}{2}(a-1)|\phi_x^0|^2 + \\ + \frac{b}{2}|\psi_x^0|^2 - \frac{1}{2}|\psi^0|^2 + \frac{1}{2}|\phi^0|^2. \end{aligned}$$

By Poincaré inequality, we get:

$$\int_0^1 |v_x(s)|^2\, ds \geq \lambda_1 \int_0^1 |v(s)|^2\, dx \quad \text{for all} \quad v \in H_0^1(0,1).$$

$\lambda_1 = \pi^2$ the first eigenvalue of $-v'' = \lambda v, \quad v \in H_0^1(\Omega)$. Then,

$$|\psi^0|^2 \leq \frac{1}{\lambda_1}|\psi_x^0|^2.$$

We obtain:

$$\frac{b}{2}|\psi_x^0|^2 - \frac{1}{2}|\psi^0|^2 \geq \frac{1}{2}\left(b - \frac{1}{\pi^2}\right)|\psi_x^0|^2$$

with $\pi^2 b > 1$ by hypothesis of Theorem 8.1.

Therefore, we modify (8.45) obtaining:

$$\begin{aligned} &E_0 + (\phi^0, \psi^0_x) \ge E_0 - \tfrac{1}{2}\left(|\psi^0|^2 + |\phi^0_x|^2\right) \ \ge \\ &\ge \tfrac{1}{2}\,|\phi^1|^2 + \tfrac{1}{2}\,|\psi^1|^2 + \tfrac{1}{2}\,(a-1)|\phi^0_x|^2 + \tfrac{1}{2}\left(b - \tfrac{1}{\pi^2}\right)|\psi^0_x|^2 + |\phi^0_x|^2. \end{aligned} \tag{8.46}$$

From (8.45) and (8.46) we get:

$$\begin{aligned} E(t) + (\psi, \phi)x) \ge \frac{1}{2}|\phi^1|^2 + \frac{1}{2}|\psi^1|^2 + \frac{1}{2}(a-1)|\phi^0_x|^2 + \\ + \frac{1}{2}\left(b - \frac{1}{\pi^2}\right)|\psi^0_x|^2 + \frac{1}{2}|\phi^0_x|^2. \end{aligned}$$

From this inequality we obtain:

$$C_0 E_0 \le E(t) \quad \text{on} \quad 0 \le t \le T. \tag{8.47}$$

Let us now prove the second member of (8.43). For this, we have:

$$E(t) + (\psi, \phi_x) = E_0 + (\psi^0, \phi^0_x) \le C\,E_0\,.$$

Since

$$-\frac{1}{2}\left(|\psi_x|^2 + |\phi|^2\right) \le -(\psi_x, \phi) = (\psi, \phi_x),$$

we obtain,

$$E(t) - \frac{1}{2}\left(|\psi_x|^2 + |\phi|^2\right) \le E(t) + (\psi, \phi_x) \le C\,E_0\,.$$

To modify the right hand side of the last inequality, we use Poincaré inequality for $|\phi|$ and obtain:

$$\begin{aligned} E(t) - \frac{1}{2}\,|\psi_x|^2 - \frac{1}{2\lambda_1}\,|\phi_x|^2 \ge \frac{1}{2}\,|\phi'|^2 + \frac{1}{2}\,|\psi'|^2 + \\ + \frac{1}{2}\left(a - \frac{1}{\lambda_1}\right)|\phi_x|^2 + \frac{1}{2}\,(b-1)|\psi_x|^2 \ge \min\left(a - \frac{1}{\lambda_1},\, b-1, 1\right)E(t). \end{aligned}$$

Then $E(t) \le C_1\,E_0$ on $0 \le t \le T$. ■

8.5 Direct and Inverse Inequalities

The key point in the proof of Theorem 8.1 was the double inequality (8.16). The right side of (8.16) is called **direct inequality** and the left one **inverse inequality**.

8.5.1 Direct Inequality

Let us consider the system (8.6) with f, g in the right hand side of $(8.6)_1$ and $(8.6)_2$, respectively, instead of zero. We shall prove first a basic identity, cf. Chapter 3. We need only $f, g \in L^2(0,T;L^2(0,1))$.

Lemma 8.1 *If $\{\phi, \psi\}$ is an weak solution of (8.6), (8.7) and (8.8) with right hand side f and g, then we have the identity:*

$$\begin{aligned}
&\frac{1}{2}\int_0^T (a\phi_x^2(0,t) + b\psi_x^2(0,t))\,dt \;= \\
&= -\left[(\phi'(x,t),(1-x)\phi_x(x,t)) + (\psi'(x,t),(1-x)\psi_x(x,t))\right]_0^T + \\
&+ \int_Q \left(|\phi'|^2 + |\psi'|^2 + a|\phi_x|^2 + b|\psi_x|^2\right) dxdt - \\
&-\frac{1}{2}\int_Q |\phi|^2\,dxdt + \int_Q f(1-x)\phi_x\,dxdt + \int_Q g(1-x)\psi_x\,dxdt.
\end{aligned} \tag{8.48}$$

Proof: The proof is done for the case $\phi^0, \psi^0 \in H_0^1(0,1) \cap H^2(0,1)$; $\phi^1, \psi^1 \in H_0^1(0,1)$ and $f, g \in W^{1,1}(0,T;H_0^1(0,1))$. Multiply $(8.20)_1$ by $(1-x)\phi_x$, $(8.20)_2$ by $(1-x)\psi_x$ and integrating on Q, we obtain:

$$\begin{aligned}
&\int_Q \phi''\cdot(1-x)\phi_x\,dxdt - a\int_Q \phi_{xx}\cdot(1-x)\phi_x\,dxdt - \\
&-\int_Q \psi_x\cdot(1-x)\phi_x\,dxdt + \int_Q \phi\cdot(1-x)\phi_x\,dxdt = \\
&\int_Q f\cdot(1-x)\phi_x\,dxdt
\end{aligned} \tag{8.49}$$

$$\begin{aligned}
&\int_Q \psi''\cdot(1-x)\psi_x\,dxdt - b\int_Q \psi_{xx}\cdot(1-x)\psi_x\,dxdt + \\
&+\int_Q \phi_x\cdot(1-x)\psi_x\,dxdt = \int_Q g\cdot(1-x)\psi_x\,dxdt
\end{aligned} \tag{8.50}$$

Let us calculate (8.49) term by term

$$\begin{aligned}
&\int_Q \phi''\cdot(1-x)\phi_x\,dxdt = \int_0^T (\phi'',(1-x)\phi_x)\,dt \;= \\
&= (\phi',(1-x)\phi_x)\Big|_0^T - \int_0^T (\phi',(1-x)\phi_x')\,dt.
\end{aligned}$$

The integral at the right hand side becomes:

$$\begin{aligned}
&\int_0^T\int_0^1 (1-x)\phi'\phi_x'\,dxdt = \frac{1}{2}\int_0^T\int_0^1 (1-x)\,\frac{\partial}{\partial x}\,|\phi'|^2\,dxdt \overset{\text{by parts}}{=} \\
&= \frac{1}{2}\int_0^T \left\{(1-x)|\phi'|^2\Big|_0^1 - \int_0^1 |\phi'|^2\,dx\right\} dt = -\frac{1}{2}\int_Q |\phi'|^2\,dxdt.
\end{aligned}$$

Note that $(1-x)\phi'(x,t)\big|_0^1 = -\phi'(0,t) = 0$. Then,

$$\int_Q \phi'' \cdot (1-x)\phi_x \, dxdt = (\phi', (1-x)\phi_x)\big|_0^T - \frac{1}{2}\int_Q \phi'(x,t)^2 \, dxdt. \tag{8.51}$$

We obtain:

$$\begin{aligned}
&-a\int_Q \phi_{xx} \cdot (1-x)\phi_x \, dxdt = -a\int_0^T \int_0^1 (1-x)\phi_x\phi_{xx} \, dxdt \; = \\
&= -\frac{a}{2}\int_0^T \int_0^1 (1-x)\,\frac{\partial}{\partial x}\,\phi_x^2 \, dxdt \;\overset{\text{by parts}}{=} \\
&= -\frac{a}{2}\int_0^T \left\{(1-x)\phi_x^2\big|_0^1 + \int_0^1 \phi_x^2 \, dx\right\} dt \; = \\
&= \frac{1}{2}\int_0^T a\phi_x^2(0,t)\, dt - \frac{1}{2}\int_Q a\phi_x^2 \, dxdt.
\end{aligned}$$

Then,

$$-\int_Q a\phi_{xx} \cdot (1-x)\phi_x \, dxdt = \frac{1}{2}\int_0^T a\phi_x^2(0,t)\, dt - \frac{1}{2}\int_Q a\phi_x^2 \, dxdt. \tag{8.52}$$

The next terms are:

$$\begin{aligned}
&-\int_Q \phi_x \cdot (1-x)\phi_x \, dxdt = -\int_Q (1-x)\phi_x\phi_x \, dxdt \; = \\
&= \int_Q \phi \cdot (1-x)\phi_x \, dxdt = \frac{1}{2}\int_Q (1-x)\,\frac{\partial}{\partial x}\,\phi^2 \, dxdt \; = \\
&= \frac{1}{2}\int_0^T \left\{(1-x)\phi^2(x,t)\big|_0^1 + \int_0^1 \phi^2 \, dx\right\} dt = \frac{1}{2}\int_Q \phi^2 \, dxdt
\end{aligned} \tag{8.53}$$

because $\phi(0,t) = 0$.

Then,

$$\int_Q \phi \cdot (1-x)\phi_x \, dxdt = -\frac{1}{2}\int_Q \phi^2 \, dxdt. \tag{8.54}$$

From (8.51) to (8.54) we obtain, by addition:

$$\begin{aligned}
&\frac{1}{2}\int_0^T a\phi_x^2(0,t)\, dt = -(\phi', (1-x)\phi_x)\big|_0^T + \\
&+\frac{1}{2}\int_Q \phi'^2 \, dxdt + \frac{1}{2}\int_Q a\phi_x^2 \, dxdt + \int_Q (1-x)\psi_x\phi_x \, dxdt - \\
&-\frac{1}{2}\int_Q \phi^2 \, dxdt + \int_Q f \cdot (1-x)\phi_x \, dxdt.
\end{aligned} \tag{8.55}$$

■

Let us now calculate (8.50). We obtain, by the same method used for (8.49):

$$\begin{aligned}\frac{1}{2}\int_0^T \psi_x^2(0,t)\,dt &= -(\psi',(1-x)\psi_x)\Big|_0^T + \\ &+\frac{1}{2}\int_Q \psi'^2\,dxdt + \frac{1}{2}\int_Q b\psi_x^2\,dxdt - \\ &-\int_Q (1-x)\psi_x\phi_x\,dxdt + \int_Q g\cdot(1-x)\psi_x\,dxdt.\end{aligned} \tag{8.56}$$

Finally, if we add (8.55) and (8.56) we obtain:

$$\begin{aligned}&\frac{1}{2}\int_0^T (a\phi_x^2(0,t)+b\psi_x^2(0,t))\,dt = \\ &= -\{(\phi',(1-x)\phi_x)+(\psi',(1-x)\psi_x)\}\Big|_0^T + \\ &+\frac{1}{2}\int_Q (|\phi'|^2+|\psi'|^2+a|\phi_x|^2+b|\psi_x|^2)\,dxdt - \\ &-\frac{1}{2}\int_Q \phi^2\,dxdt + \int_Q f\cdot(1-x)\phi_x\,dxdt + \int_Q g\cdot(1-x)\psi_x\,dxdt.\end{aligned}$$

■

Now, using the identity (8.48) of Lemma 8.1, we are able to prove the direct inequality. By limits, the identity is true for weak solution. Note that:

$$\begin{aligned}&|(\phi',(1-x)\phi_x)|\Big|_0^T \le 2\sup_{0\le t\le T}|(\phi'(x,t),(1-x)\phi_x(x,t)| \le \\ &\le 2\sup_{0\le t\le T}\left\{\frac{1}{2}\int_0^1 \phi'^2(x,t)\,dx + \frac{1}{2}\int_0^1 \phi_x^2(x,t)\,dx\right\} \le C_0\,E_0\,.\end{aligned}$$

■

To prove the direct inequality, let us consider (8.48) with $f=g=0$, that is, $\{\phi,\psi\}$ is a solution of (8.6), (8.7) and (8.8). If we consider the inequality $C_0E_0 \le E(t) \le C_1E_0$ and Poincaré inequality, it follows from (8.48) that:

$$\int_0^T (a\phi_x^2(0,t)+b\psi_x^2(0,t))\,dt \le C\left\|\{\phi^0,\phi^1,\psi^0,\psi^1\}\right\|^2_{(H_0^1(0,1)\times L^2(0,1))^2}\,.$$

■

8.5.2 Inverse Inequality

We will prove the inverse inequality following the method of Zuazua [71]. In fact, let us consider the functional:

$$\begin{aligned}F(x) &= \frac{1}{2}\int_{\alpha x}^{T-\alpha x} (\phi'(x,t)^2 + a\phi_x(x,t)^2 + \phi(x,t)^2)\,dt + \\ &+\frac{1}{2}\int_{\alpha x}^{T-\alpha x} (\psi'(x,t)^2 + b\psi_x(x,t)^2)\,dt,\end{aligned} \tag{8.57}$$

defined on $0 \le x \le 1$. When $x = 0$ we have:

$$F(0) = \frac{1}{2}\int_0^T \left(a\phi_x(0,t)^2 + b\psi_x(0,t)^2\right) dt. \tag{8.58}$$

which is the second hand side of the inverse inequality. Note that $\alpha = \max\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}\right)$.
Represent $F(x) = G(x) + H(x)$ given by (8.57). Taking the derivative of $F(x)$ we have:

$$\begin{aligned} G'(x) = &\int_{\alpha x}^{T-\alpha x} (\phi'\phi'_x + a\phi_x\phi_{xx} + \phi\phi_x)\, dt - \\ -&\frac{\alpha}{2}\sum_{\substack{t=T-\alpha x \\ t=\alpha x}} \left(\phi'(x,t)^2 + a\phi_x(x,t)^2 + \phi(x,t)^2\right). \end{aligned} \tag{8.59}$$

Integrating by parts:

$$\int_{\alpha x}^{T-\alpha x} \phi'\phi'_x\, dt = \phi'\phi_x\Big|_{\alpha x}^{T-\alpha x} - \int_{\alpha x}^{T-\alpha x} \phi''\phi_x\, dt. \tag{8.60}$$

Multiply both sides of (8.6) by ϕ_x and integrate on $(\alpha x, T-\alpha x)$ with respect to t. We get

$$\begin{aligned} &\int_{\alpha x}^{T-\alpha x} \phi''\phi_x\, dt - \int_{\alpha x}^{T-\alpha x} a\phi_{xx}\phi_x\, dt - \\ -&\int_{\alpha x}^{T-\alpha x} \psi_x\phi_x\, dt + \int_{\alpha x}^{T-\alpha x} \phi\phi_x\, dt = 0. \end{aligned} \tag{8.61}$$

Substituting (8.57) in (8.60) we get:

$$\begin{aligned} &\phi'\phi_x\Big|_{\alpha x}^{T-\alpha x} - \int_{\alpha x}^{T-\alpha x} \phi'\phi'_x\, dt - \int_{\alpha x}^{T-\alpha x} a\phi_{xx}\phi_x\, dt - \\ -&\int_{\alpha x}^{T-\alpha x} \psi_x\phi_x\, dt + \int_{\alpha x}^{T-\alpha x} \phi\phi_x\, dt = 0. \end{aligned} \tag{8.62}$$

Adding (8.59) and (8.62) we have:

$$\begin{aligned} G'(x) = &\phi'\phi_x\Big|_{\alpha x}^{T-\alpha x} - \int_{\alpha x}^{T-\alpha x} \psi_x\phi_x\, dt + 2\int_{\alpha x}^{T-\alpha x} \phi\phi_x\, dt - \\ -&\frac{\alpha}{2}\sum_{\substack{t=T-\alpha x \\ t=\alpha x}} \left(\phi'(x,t)^2 + a\phi_x(x,t)^2 + \phi^2(x,t)\right). \end{aligned} \tag{8.63}$$

The first term on the right hand side of (8.63) can be dominated as follows:

$$\phi'\phi_x \le \frac{\beta}{2}\,\phi'^2 + \frac{1}{2\beta}\,\phi_x^2 = \frac{\beta}{2}\,\phi'^2 + \frac{1}{2\beta a}\,a\phi_x^2\,.$$

Taking $\beta = \frac{1}{\sqrt{a}}$ we get:

$$\phi'\phi_x \le \frac{1}{\sqrt{a}}\left(\frac{1}{2}\,\phi'^2 + \frac{1}{2}\,a\phi_x^2\right).$$

We know, by hypothesis of Theorem 8.1, that $\frac{1}{\sqrt{a}} \le \alpha$, then

$$\phi'\phi_x \le \frac{\alpha}{2}\left(\phi'(x,t)^2 + a\phi_x(x,t)^2\right),$$

whence,

$$\phi'\phi_x\Big|_{\alpha x}^{T-\alpha x} \le \frac{\alpha}{2}\sum_{\substack{t=T-\alpha x\\ t=\alpha x}}\left(\phi'(x,t)^2 + a\phi_x(x,t)^2 + \phi(x,t)^2\right. \tag{8.64}$$

From (8.63) and (8.64) we obtain:

$$G'(x) \le -\int_{\alpha x}^{T-\alpha x}\psi_x\phi_x\,dt + 2\int_{\alpha x}^{T-\alpha x}\phi\phi_x\,dt. \tag{8.65}$$

The derivative of $H(x)$ with respect to x is

$$H'(x) = \int_{\alpha x}^{T-\alpha x}\left(\psi'\psi'_x + b\psi_x\psi_{xx}\right)dt - \sum_{\substack{t=T-\alpha x\\ t=\alpha x}}\left(\psi'(x,t)^2 + b\psi_x(x,t)^2\right). \tag{8.66}$$

By the same argument used in the analysis of $G'(x)$ we obtain from (8.66):

$$H'(x) = \psi'\psi_x\Big|_{\alpha x}^{T-\alpha x} + \int_{\alpha x}^{T-\alpha x}\phi_x\psi_x\,dt - \frac{\alpha}{2}\sum_{\substack{t=T-\alpha x\\ t=\alpha x}}\left(\psi'(x,t)^2 + b\psi_x(x,t)^2\right). \tag{8.67}$$

We also obtain:

$$\psi'\psi_x\Big|_{\alpha x}^{T-\alpha x} \le \frac{\alpha}{2}\sum_{\substack{t=T-\alpha x\\ t=\alpha x}}\left(\psi'(x,t)^2 + b\psi_x(x,t)^2\right). \tag{8.68}$$

From (8.67) and (8.68) we obtain:

$$H'(x) \le \int_{\alpha x}^{T-\alpha x}\phi_x\psi_x\,dt. \tag{8.69}$$

Adding (8.65) and (8.69) we get:

$$F'(x) \le 2\int_{\alpha x}^{T-\alpha x}\phi\phi_x\,dt; \tag{8.70}$$

Since

$$\phi\phi_x \le \max\left\{1, \frac{1}{a}\right\}\left(\frac{1}{2}\,\phi(x,t)^2 + \frac{1}{2}\,a\phi_x(x,t)^2\right),$$

we modify (8.70) to obtain:

$$F'(x) \le C\,F(x). \tag{8.71}$$

Integrating (8.71) we have:

$$F(x) \le e^c\,F(0)$$

or

$$\int_0^1 F(x)\,dx \le e^c\,F(0).$$

Since $T > 2\alpha$, we obtain:

$$\begin{aligned}(T-2\alpha)E_0 = \int_{\alpha x}^{T-\alpha x} E_0\,dt \le C_1 \int_{\alpha}^{T-\alpha} E(t)\,dt \; = \\ = C_1 \int_{\alpha}^{T-\alpha}\int_0^1 \left(\frac{1}{2}\,\phi'^2 + \frac{1}{2}\,a\phi_x^2 + \frac{1}{2}\,\phi^2\right) dxdt + \\ + C_1 \int_{\alpha}^{T-\alpha}\int_0^1 \left(\frac{1}{2}\,\psi'^2 + \frac{1}{2}\,b\psi_x^2\right) dxdt.\end{aligned}$$

Since $0 \le x \le 1$, we obtain:

$$\begin{aligned}(T-2\alpha)E_0 \le C_1 \int_{\alpha x}^{T-\alpha x}\int_0^1 \left(\frac{1}{2}\,\phi'^2 + \frac{1}{2}\,a\phi_x^2 + \frac{1}{2}\,\phi^2\right) dxdt + \\ + C_1 \int_{\alpha x}^{T-\alpha x}\int_0^1 \left(\frac{1}{2}\,\psi'^2 + \frac{1}{2}\,b\psi_x^2\right) dxdt = C_1 \int_0^1 F(x)\,dx \le C_2\,F(0).\end{aligned}$$

Then for $T < 2\alpha$ we obtain the inverse inequality and consequently the proof of Theorem 8.1. ■

8.6 Non Homogeneous Mixed Problem for the Timoshenko System. Ultra Weak Solutions.

We consider, now, the non homogeneous mixed problem:

$$\left|\begin{aligned} & y'' - ay_{xx} - z_x + y = 0 \quad \text{in} \quad Q, \\ & z'' - bz_{xx} + y_x = 0 \quad \text{in} \quad Q, \end{aligned}\right. \tag{8.72}$$

$$\left|\begin{aligned} & y(0,T) = v(t),\; y(1,t) = 0 \quad \text{in} \quad]0,T[\,, \\ & z(0,T) = w(t),\; z(1,t) = 0 \quad \text{in} \quad]0,T[\,. \end{aligned}\right. \tag{8.73}$$

$$\left|\begin{aligned} & y(x,0) = y^0(x),\; y'(x,0) = y^1(x) \quad \text{in} \quad]0,T[\,, \\ & z(x,0) = z^0(x),\; z'(x,0) = z^1(x) \quad \text{in} \quad]0,T[\,. \end{aligned}\right. \tag{8.74}$$

We want to study this problem when we suppose $v(t)$, $w(t)$ in $L^2(0,T)$. In order to obtain the definition of solution for the above mixed problem, we follows an heuristic procedure. Multiply both sides of $(8.72)_1$ by ϕ and of $(8.72)_2$ by ψ and integrate on Q. Here $\{\phi,\psi\}$ is the solution of:

$$\left|\begin{aligned} & \phi'' - a\phi_{xx} - \psi_x + \phi = f \quad \text{in} \quad Q, \\ & \psi'' - b\psi_{xx} + \phi_x = g \quad \text{in} \quad Q. \end{aligned}\right. \tag{8.75}$$

$$\left|\begin{array}{l}\phi(0,t)=0,\ \phi(1,t)=0 \quad \text{in} \quad]0,T[\,,\\ \psi(0,t)=0,\ \psi(1,t)=0 \quad \text{in} \quad]0,T[\,,\end{array}\right. \tag{8.76}$$

$$\left|\begin{array}{l}\phi(x,T)=0,\ \phi'(x,T)=0 \quad \text{in} \quad]0,T[\,,\\ \psi(x,T)=0,\ \psi'(x,T)=0 \quad \text{in} \quad]0,T[\,.\end{array}\right. \tag{8.77}$$

We know, that the solution $\{\phi,\psi\}$ belongs to the class

$$\phi,\psi\in C^0\big([0,T];H_0^1(0,1)\big)\cap C^1\big([0,T].L^2(0,1)\big),$$

where $f,g\in L^1(0,T;L^2(0,1))$.

After the integration on Q, we obtain:

$$\begin{aligned}&-(y^1,\phi(0))+(y^0,\phi'(0))-(z^1,\psi(0))+(z^0,\psi'(0))-\\&-\int_0^T av(t)\phi_x(0,t)\,dt-\int_0^T bw(t)\phi_x(0,t)\,dt+\\&+\int_0^T\int_0^1(yf+zg)\,dxdt=0.\end{aligned} \tag{8.78}$$

Note that $\phi(0),\psi(0)\in H_0^1(0,1)$ and $\phi'(0),\psi'(0)\in L^2(0,1)$, therefore, if in (8.78) we choose $y^0,z^0\in L^2(0,1)$ and $y^1,z^1\in H^{-1}(0,1)$, then make sense, in (8.78), $\langle y^1,\phi(0)\rangle$, $\langle z^1,\psi(0)\rangle$, duality pairing between $H^{-1}(0,1)$ and $H_0^1(0,1)$. Also make sense $(y^0,\phi'(0))$, $(z^0,\psi'(0))$ the inner product in $L^2(0,1)$. Note that $\phi_x(0,t)$, $\psi_x(0,t)$ belong to $L^2(0,1)$, cf. §4, inequalities.

Motivated by (8.78) we consider the map S defined on $\big(L^1(0,T;L^2(\Omega))\big)^2$, with real values, by:

$$\begin{aligned}&\langle S,\{f,g\}\rangle=(y^0,\phi'(0))-\langle y^1,\phi(0)\rangle+(z^0,\psi'(0))-\langle z^1,\psi(0)\rangle+\\&+\int_0^T av(t)\phi_x(0,t)\,dt+\int_0^T bw(t)\psi_x(0,t)\,dt.\end{aligned} \tag{8.79}$$

Whence

$$\begin{aligned}&|\langle S,\{f,g\}\rangle|\le|y^0|\,|\phi'(0)|+\|y^1\|_{H^{-1}(0,1)}\ ||\phi(0)||+\\&+|z^0|\,|\psi'(0)|+\|z^1\|_{H^{-1}(0,1)}\ ||\psi(0)||+\\&+a|v(t)|\,|\phi_x(0,t)|+b|w(t)|\,|\psi_x(0,t)|.\end{aligned} \tag{8.80}$$

As a consequence of the Theorem 8.2 and the identity of Lemma 8.1, we obtain, from (8.80):

$$\begin{aligned}&|\langle S,\{f,g\}\rangle|\le C\big(|y^0|+\|y^1\|_{H^{-1}(0,1)}+|z^0|+\|z^1\|_{H^{-1}(0,1)}+\\&+|\phi_x(0,t)|+|\psi_x(0,t)|\big)\ \|\{f,g\}\|_{(L^1(0,T;L^2(0,1)))^2}\ .\end{aligned} \tag{8.81}$$

Then (8.81) says that S defined by (8.79) on $\big(L^1(0,T;L^2(0,1))\big)^2$ is a continuous linear form, that is, S is an object of $\big(L^\infty(0,T;L^2(0,1))\big)^2$ dual of $\big(L^1(0,T;L^2(0,1))\big)^2$. By Riesz's representation theorem, exists an object $\{y,z\}$ of $\big(L^\infty(0,T;L^2(0,1))\big)^2$ such that:

$$\langle S\{f,g\}\rangle = \int_0^T\int_0^1 (yf+zg)\,dxdt. \tag{8.82}$$

Definition 8.1 *For $\{y^0,y^1\},\{z^0,z^1\} \in L^2(0,1)\times H^{-1}(0,1)$ and $v,w\in L^2(0,T)$, we call ultra weak solution or solution by transposition of the non homogeneous problem (8.72), (8.73) and (8.74), the pair of functions $\{y,z\}\in \big(L^\infty(0,T;L^2(0,1))\big)^2$ such that satisfies:*

$$\begin{aligned}\int_0^T\!\!\int_0^1 (yf+zg)\,dxdt &= (y^0,\phi(0)) - \langle y^1,\phi(0)\rangle + (z^0,\psi'(0)) - \\ &- \langle z^1,\psi(0)\rangle + \int_0^T av(t)\phi_x(0,t)\,dt + \int_0^T bw(t)\phi_x(0,t)\,dt,\end{aligned}$$

for all pair $\{f,g\}\in \big(L^1(0,T;L^2(0,1))\big)^2$ and $\{\phi,\psi\}$ is solution of (8.75), (8.76) and (8.77).

Exists only one ultra weak solution for the Timoshenko system and this solution $\{y,z\}$ satisfies:

$$\begin{aligned}&\|\{y,z\}\|_{(L^\infty(0,T;L^2(0,1)))^2} \le \\ &\le C\big(|y^0| + \|y^1\|_{H^{-1}(0,1)} + |z^0| + \|z^1\|_{H^{-1}(0,1)} + |\phi_x(0,t)| + |\psi_x(0,t)|\big).\end{aligned} \tag{8.83}$$

In fact. the existence is a consequence of Riesz's representation theorem, as we have seen above. The estimate (8.83) is a consequence of (8.81). The uniqueness follows from Du Bois Raymond's Lemma.

■

We can also prove using the same method as in Chapter 4 and 5 that the ultra weak solution $\{y,z\}$ of (8.72), (8.73) and (8.74) satisfies the regularity condition:

$$y,z\in C^0\big([0,T];L^2(0,1)\big)\cap C^1\big([0,T];H^{-1}(0,1)\big),$$

the initial data and the boundary conditions. ■

Chapter 9

HUM and the Wave Equation with Variable Coefficients

9.1 Introduction.

Let Ω be a bounded domain $\mathbb{R}^n$ with boundary Γ and Q the finite cylinder $Q = \Omega\times]0, T[$ with lateral boundary $\Sigma = \Gamma\times]0, T[$. We consider the following system:

$$\left| \begin{array}{l} u'' - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u'}{\partial x_i} + \sum_{i=1}^{n} d_i(x,t)\frac{\partial u}{\partial x_i} = 0 \text{ in } Q, \\ u = v \text{ in } \Sigma = \Gamma\times]0,T[, \\ u(0) = u^0,\ u'(0) = u^1 \text{ in } \Omega \end{array} \right. \qquad (*)$$

where u' stands for $\frac{\partial u}{\partial t}$ and $u(0)$, $u'(0)$ denote, respectively, the functions $x \mapsto u(x,0)$, $x \mapsto u'(x,0)$. Here v is the control variable, that is, we act on the system $(*)$ through the lateral boundary Σ.

The problem for exact controllability of system $(*)$ states as follows: Given $T > 0$ large enough, is it possible, for every initial data $\{u^0, u^1\}$ lie in an appropriate space on Ω to find a corresponding control v driving the system to rest at time T, i.e., such that the solution $u(x,t)$ of $(*)$ satisfies

$$u(T) = 0,\ u'(T) = 0?$$

1

[1] This part is a paper that was published for one of Authors in Asymptotic Analysis 11 (1995), pp. 317-341.

System $(*)$ is motivated in the study of the boundary exact controllability for the wave equations in $\widehat{Q}$, $\widehat{Q}$ a particular non-cylindrical domain. A particular system $(*)$ appears when the wave equation $\widehat{u}'' - \triangle\widehat{u} = 0$ defined in $\widehat{Q}$ is transformed in a equation defined in Q, as we shall see in the following chapter. Our objective is to show that this is particular system is exact controllable. For that we use the Hilbert Uniqueness Method (HUM) introduced by J. L. Lions [8] and [10]. This is possible because in this case we have uniqueness, reversibility and smoothness of solutions.

Concerning to the exact controllability for system $(*)$ we note that the case $a_{ij} = \delta_{ij}a(t)$, $b_i = d_i = 0$ was studied by J.L. Lions [36] and the case $a_{ij} = \delta_{ij}a(x)$, $b_i = d_i = 0$, by E. Zuazua [69]. Also, R. Fuentes [16] analysed the situation $b_i = d_i = 0$. Our approach is different of this one and we note that the presence of the term $\frac{\partial u'}{\partial x_i}$ in $(*)$ gives hard technical difficulties.

A number of authors have used the HUM in the study of exact controllability of distributed system among of them we can mention J.P.Puel [17], J.P.Puel and E. Zuazua [58], C. Fabre [11], C. Fabre and J.P.Puel [12], E. Zuazua [68], [69], V. Komornik [26] and L.A. Medeiros [47].

This chapter is organized as follows:

- Main result.
- The Homogeneous Problem.
- Inverse and Direct Inequality.
- Exact Controllability.

9.2 Main Result

Let us introduce some notations (cf. J.L. Lions [40]). Let $x^0 \in \mathbb{R}^n$, $m(x) = x - x^0$ and $\nu(x)$ the unit normal vector at $x \in \Gamma$, directed towards the exterior of Ω. Consider the sets

$$\Gamma(x^0) = \{x \in \Gamma; m(x)\cdot\nu(x) \geq 0\},\ \Gamma_*(x^0) = \Gamma \setminus \Gamma(x^0),\ \Sigma(x^0) = \Gamma(x^0)\times]0,T[.$$

In the definition of $\Gamma(x^0)$, $\cdot$ denotes the scalar product in $\mathbb{R}^n$. We consider:

$$R(x^0) = \sup_{x\in\Omega}|m(x)|, \quad M = \sup_{x\in\Omega}|x|$$

and λ_1 the first eigenvalue of the spectral problem $-\triangle\varphi = \lambda\varphi$, $\varphi \in H_0^1(\Omega)$. Let $k : [0,\infty[\to [0,\infty[$ be a continuous function. All scalar functions considered in the problem will be real-valued.

We make the following assumptions:

$$\Omega \text{ contains the origin of } \mathbb{R}^n; \tag{H1}$$

(This hypothesis is introduced in order to facilitate the computations but it is not necessary);

$$\text{The boundary } \Gamma \text{ of } \Omega \text{ is } C^2; \tag{H2}$$

and concerning the function k,

$$k \in W^{3,\infty}_{loc}(]0,\infty[), \tag{H3}$$

$$0 < k_0 = \inf_{t\geq 0} k(t),\ \sup_{t\geq 0} k(t) = k_1 < \infty, \tag{H4}$$

$$\sup_{t\geq 0} |k'(t)| = \tau < \frac{1}{M}, \tag{H5}$$

$$l_1 = \int_0^\infty |k'(t)|dt < \infty,\ l_2 = \int_0^\infty |k''(t)|dt < \infty\ . \tag{H6}$$

We consider the operator

$$\left| \begin{array}{l} Lu = u'' - \dfrac{\partial}{\partial x_i}\left[(\delta_{ij} - k'^2 x_i x_j)k^{-2}\dfrac{\partial u}{\partial x_j}\right] - 2k'k^{-1}x_i\dfrac{\partial u'}{\partial x_i} + \\ + [(1-n)k'^2 - k''k]k^{-2}x_i\dfrac{\partial u}{\partial x_i} \end{array} \right. \tag{9.1}$$

where δ_{ij} is the Kronecker delta.

Remark 9.1 *Here and in what follows the summation convention of repeated indices is adopted.*

The formal adjoint L^* of L is

$$\left| \begin{array}{l} L^*z = z'' - \dfrac{\partial}{\partial x_i}\left[(\delta_{ij} - k'^2 x_i x_j)k^{-2}\dfrac{\partial z}{\partial x_j}\right] - 2k'k^{-1}\dfrac{\partial z'}{\partial x_i} - \\ - 2nk'k^{-1}z' + [(n+1)k'^2 - k''k]k^{-2}x_i\dfrac{\partial z}{\partial x_i} + \\ + [n(n+1)k'^2 - nk''k]k^{-2}z \end{array} \right. \tag{9.2}$$

We want to act on only a part of the boundary Σ, more precisely, one considers the following system:

$$\left| \begin{array}{l} Lu = 0 \text{ in } Q, \\ u = \begin{cases} v & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma \setminus \Sigma(x^0), \end{cases} \\ u(0) = u^0,\ u'(0) = u^1 \text{ in } \Omega. \end{array} \right. \tag{9.3}$$

Remark 9.2 *In Remark 9.8 we will give a special time T_0 depending on n, $R(x^0)$, λ_1, the function k and on the geometry of Ω.*

Now we states the main result of the chapter.

Theorem 9.1 *We assume that hypotheses $(H1)$-$(H6)$ are satisfied. Let $T > T_0$. Then for every initial data $\{u^0, u^1\}$ belonging to $L^2(\Omega)\times H^{-1}(\Omega)$, there exists a control $v \in L^2(\Sigma(x^0))$ such that the solution defined by transposition u of Problem (9.3) satisfies*

$$u(T) = 0, u'(T) = 0.$$

Remark 9.3 *By applying the same arguments used in the of the proof of Theorem 9.1 we obtain the exact controllability for system $(*)$ with $b_i = d_i = 0$ and $a_{ij}(x,t)$ in the hypotheses of R. Fuentes, loc. cit, that is, the $a'_{ij}s$ are smooth, symmetric, uniformly elliptic on Q,*

$$\sup_{x\in\Omega}\int_0^\infty |a'_{ij}(x,t)|dt < \infty$$

and there exists $\delta > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j - \frac{1}{2}\frac{\partial}{\partial x_l}a_{ij}(x,t)m_l(x)\xi_i\xi_j \geq \delta a_{ij}(x,t)\xi_i\xi_j$$

for all $(x,t) \in Q$ and $\xi \in \mathbb{R}^n$.

The proof of Theorem 9.1 will be done in the next three sections.

9.3 The Homogeneous Problem

Let us introduce some notations that it will be used in what follows. With $(\cdot,\cdot)$, $|\cdot|$ we will denote the inner product and norm of $L^2(\Omega)$ and with $\|\cdot\|$, the norm of $H_0^1(\Omega)$ given by the Dirichlet form. The duality pairing between the space F and its dual F' will be noted by $\langle\cdot,\cdot\rangle$.

In this section we obtain the existence and the identity of energy of solutions of a mixed problem the general operator of second order in t.

$$Ru = u'' + A(t)u + b_i(x,t)\frac{\partial u'}{\partial x_i} + c(x,t)u' + d_i(x,t)\frac{\partial u}{\partial x_i} + f(x,t)u \tag{9.4}$$

where

$$A(t)u = -\frac{\partial u}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_i}\right) \tag{9.5}$$

The coefficients a_{ij} satisfy the following hypotheses:

$$\begin{cases} a_{ij} \text{ are symmetric and uniformly coercive on } \; Q; \\ a_{ij} \in C^1(\overline{Q}),\; a''_{ij} \in L^\infty(Q); \\ b_i, c, d_i, f \in W^{1,\infty}(0,T;L^\infty(\Omega));\; \dfrac{\partial b_i}{\partial x_i} \in L^\infty(Q). \end{cases} \tag{9.6}$$

Let us consider the problem

$$\left| \begin{array}{l} Ru = h \text{ in } Q, \\ u = 0 \text{ in } \Sigma, \\ u(0) = u^0,\; u'(0) = u^1 \text{ in } \Omega \end{array} \right. \tag{9.7}$$

with data $\{u^0, u^1, h\} \in H_0^1(\Omega) \times L^2(\Omega) \times L^1(0,T;L^2(\Omega))$. A function $u : Q \to \mathbb{R}$ will be called a weak solution of Problem (9.7) if u belongs to the class

$$u \in L^\infty(0,T;H_0^1(\Omega)),\; u' \in L^\infty(0,T;L^2(\Omega)),$$

satisfies the equation

$$\left| \begin{array}{l} -\displaystyle\int_0^T (u', \xi')dt + \int_0^T a(t,u,\xi)dt + \int_0^T \left\langle b_i \frac{\partial u'}{\partial x_i}, \xi \right\rangle dt + \\ + \displaystyle\int_0^T (Pu, \xi)dt = \int_0^T (h, \xi)dt,\; \forall \xi \in L^\infty(0,T;H_0^1(\Omega)), \\ \xi' \in L^2(0,T;L^2(\Omega)),\; \xi(0) = \xi(T) = 0, \end{array} \right. \tag{9.8}$$

and the initial conditions

$$u(0) = u^0,\; u'(0) = u^1.$$

Here,

$$\langle A(t)u, \xi \rangle = a(t,u,\xi) = \int_\Omega a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \xi}{\partial x_i} dx \tag{9.9}$$

and

$$Pu = cu' + d_i \frac{\partial u}{\partial x_i} + fu. \tag{9.10}$$

Theorem 9.2 *Let*

$$u^0 \in H^2(\Omega) \cap H_0^1(\Omega);\; u^1 \in H_0^1(\Omega); h, h' \in L^1(0,T;L^2(\Omega)).$$

Then there exists a unique weak solution u of Problem (9.7) in the class

$$u \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega)),\; u' \in L^\infty(0,T;H_0^1(\Omega)),\; u'' \in L^\infty(0,T;L^2(\Omega)).$$

Theorem 9.2 is showed by applying the Galerkin method with two estimates and the below remark.

Remark 9.4 *The Green's formula gives*

$$(i)\qquad \left(b_i\frac{\partial\xi}{\partial x_i},\xi\right)=-\frac{1}{2}\left(\frac{\partial b_i}{\partial x_i}\xi,\xi\right),\ \ \xi\in H_0^1(\Omega).$$

We have

$$(ii)\qquad \left|\begin{array}{l} a'(t,u,u'')=\frac{d}{d}a'(t,u,u')-a''(t,u,u')-a'(t,u',u'),\\ |a'(t,u(t),u''(t))|\le\frac{C}{\eta}\|u(t)\|^2+\eta\|u'(t)\|^2\end{array}\right.$$

where η is an arbitrary positive constant. Theorem 9.2 permits to obtain the following result:

Theorem 9.3 *Let*

$$u^0\in H_0^1(\Omega),\ \ u^1\in L^2(\Omega),h\in L^1(0,T;L^2(\Omega)).$$

Then

(i) There exists a unique weak solution u of Problem (9.7) belonging to the class

$$u\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega)).$$

(ii) The linear application

$$H_0^1(\Omega)\times L^2(\Omega)\times L^1(0,T;L^2(\Omega))\mapsto C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega)),$$
$$\{u^0,u^1,h\}\mapsto u$$

is continuous, u obtained in (i).

(iii) The solution u found in (i) satisfies

$$\frac{1}{2}|u'(t)|^2+\frac{1}{2}a(t,u(t),u(t))=\frac{1}{2}|u^1|^2+\frac{1}{2}a(0,u^0,u^0)+\frac{1}{2}\int_0^t a'(s,u,u)ds+$$
$$+\int_0^t(h,u')ds+\frac{1}{2}\int_0^t\left(\frac{\partial b_i}{\partial x_i}u',u'\right)ds-\int_0^t(Pu,u')ds,$$

where Pu was defined in (9.10).

Proof. Let (u_μ^0), (u_μ^1), (h_μ) be sequence of vectors of $H^2(\Omega)\cap H_0^1(\Omega)$, $H_0^1(\Omega)$ and $W^{1,1}(0,T;L^2(\Omega))$, respectively, such that

$$\begin{aligned} &u_\mu^0\to u^0\text{ in }H_0^1(\Omega),\ \ u_\mu^1\to u^1\text{ in }L^2(\Omega),\\ &h_\mu\to h\text{ in }L^1(0,T;L^2(\Omega)).\end{aligned}\tag{9.11}$$

Denote by u_μ a solution obtained in Theorem 9.2 with data u_μ^0, u_μ^1, h_μ. Then we have

$$u_\mu\in C(0,T;H_0^1(\Omega))\cap C^1(0,T;L^2(\Omega)).$$

Developing $(Ru_\mu, u'_\mu) = (h, u'_\mu)$, using Remark 9.4, part (i) and

$$a(t,u,u') = \frac{1}{2}\frac{d}{dt}a(t,u,u) - \frac{1}{2}a'(t,u,u)$$

we obtain identity (iii) with u_μ, more precisely,

$$\begin{aligned} E_\mu(t) = E_\mu(0) + \frac{1}{2}\int_0^t a'(s,u_\mu,u_\mu)ds + \int_0^t (h_\mu, u'_\mu)ds + \\ + \frac{1}{2}\int_0^t \left(\frac{\partial b_i}{\partial x_i}u'_\mu, u'_\mu\right)ds - \int_0^t (Pu_\mu, u'_\mu)ds, \end{aligned} \tag{9.12}$$

where

$$E_\mu(t) = \frac{1}{2}|u'_\mu(t)|^2 + \frac{1}{2}a(t, u_\mu(t), u_\mu(t)).$$

(We can also obtain identity (9.12), since u_μ is regular, applying the energy identity of Lions and Magenes [43], p. 298).

Using the smooth conditions (9.6) on the coefficients of R in (9.12), we get

$$E_\mu(t) \le E_\mu(0) + \int_0^t |h_\mu||u'_\mu|ds + C\int_0^t E_\mu(s)ds$$

hence, by Gronwall inequality,

$$E_\mu(t) \le \left[E_\mu(0) + \left(\int_0^T |h_\mu|dt\right)^2\right]e^{CT} \tag{9.13}$$

where C is a constant independent of μ and $t \in [0,T]$.

Clearly, if we repeat the arguments used in the notation of (9.13) with $u_\mu - u_\sigma$ instead of u_μ, we obtain

$$\begin{aligned} &|u'_\mu(t) - u'_\sigma(t)|^2 + a(t, u_\mu(t) - u_\sigma)(t), u_\mu(t) - u_\sigma(t)) \le \\ &\le 2\left[|u^1_\mu - u^1_\sigma|^2 + a(0, u^0_\mu - u^0_\sigma, u^0_\mu - u^0_\sigma) + \left(\int_0^T |h_\mu - h_\sigma|dt\right)^2\right]e^{CT} \end{aligned} \tag{9.14}$$

Taking the limit in this expression and considering the convergences (9.11), we find a function u such that

$$\begin{aligned} &u_\mu \to u \text{ in } C([0,T]; H^1_0(\Omega)); \\ &u'_\mu \to u' \text{ in } C([0,T]; L^2(\Omega)). \end{aligned} \tag{9.15}$$

These convergences are sufficient to complete the proof of theorem, except uniqueness. In fact, if we take the limit in (9.8),(writing with u_μ instead of u,) in (9.11) and using (9.12), we obtain (i), (ii), (iii) of the theorem. The uniqueness is proved by using a method due to M.I. Visik and O.A. Ladyzhenskaja [19] (see also [12]).

■

Next we consider a problem that will be used in the study of regularity for the solution u of Problem $(*)$ of the Introduction. This is,

$$\left|\begin{array}{l} Ru = h' \text{ in } Q, \\ u = 0 \text{ in } \Sigma, \\ u(0) = 0, \;\; u'(0) = 0 \text{ in } \Omega. \end{array}\right. \tag{9.16}$$

The weak solution u of this problem has the regularity (i) of Theorem 9.2 if $h' \in L^1(0,T;L^2(\Omega))$. We have the following estimate:

Theorem 9.4 *Let*

$$h \in L^2(0,T;H_0^1(\Omega)), \;\; h' \in L^2(0,T;L^2(\Omega)), \;\; h(0) = 0.$$

Then the solution u of Problem (9.16) satisfies

$$\|u(t)\| + |u'(t) - h(t)| \leq C \int_0^T \|h\| dt, \;\; \forall t \in [0,T], \tag{9.17}$$

where C is a constant independent of u and h.

Proof: Theorem 9.3 gives

$$\begin{aligned} &\frac{1}{2}|u'(t)|^2 + \frac{1}{2}a(t,u(t),u(t)) = \frac{1}{2}\int_0^t a'(t,u,u)ds + \int_0^t (h',u')ds + \\ &+ \frac{1}{2}\int_0^t \left(\frac{\partial b_i}{\partial x_i}u', u'\right) ds - \int_0^t (Pu,u')ds \end{aligned} \tag{9.18}$$

where Pu was defined in (9.10).

By integration by parts on $[0,t]$ and noting that $h(0) = 0$, we have

$$\int_0^t (h',u')ds = (h(t),u'(t)) - \int_0^t (h,u'')ds$$

hence

$$\begin{aligned} &\int_0^t (h',u')ds = (h(t),u'(t)) - \frac{1}{2}|h(t)|^2 + \int_0^t (h,Au)ds + \\ &+ \int_0^t \left(h, b_i\frac{\partial u'}{\partial x_i}\right) ds + \int_0^t (h,Pu)ds \end{aligned} \tag{9.19}$$

because

$$u'' = h' - Au - b_i\frac{\partial u'}{\partial x_i} - Pu.$$

Combining (9.18) and (9.19), and noting that

$$\int_0^t \left(h, b_i\frac{\partial u'}{\partial x_i}\right) ds = -\int_0^t \left(h, \frac{\partial b_i}{\partial x_i}u'\right) ds - \int_0^t \left(\frac{\partial h}{\partial x_i}, b_i u'\right) ds$$

one has

$$\begin{aligned}
&\frac{1}{2}|u'(t)-h(t)|^2+\frac{1}{2}a(t,u(t),u(t))=\frac{1}{2}\int_0^t a'(s,u,u)ds+\\
&+\int_0^t\left(\frac{\partial b_i}{\partial x_i}u',u'\right)ds+\int_0^t(h,Au)ds-\int_0^t\left(h,\frac{\partial b_i}{\partial x_i}u'\right)ds-\\
&-\int_0^t\left(\frac{\partial h}{\partial x_i},b_iu'\right)ds+\int_0^t(h,Pu)ds-\int_0^t(Pu,u')ds.
\end{aligned}\tag{9.20}$$

Making $\theta=u'-h$ in (9.20) and substituting u' by $\theta+h$ in this equality, we obtain after direct computations

$$\begin{aligned}
&\frac{1}{2}|\theta(t)|^2+\frac{1}{2}a(t,u(t),u(t))=\frac{1}{2}\int_0^t a'(s,u,u)ds+\\
&+\int_0^t(h,Au)ds+\frac{1}{2}\int_0^t\left(\frac{\partial b_i}{\partial x_i}\theta,\theta\right)ds-\int_0^t\left(\frac{\partial h}{\partial x_i},b_i\theta\right)ds-\\
&-\int_0^t(c\theta,\theta)ds-\int_0^t(ch,\theta)ds-\int_0^t\left(d_i\frac{\partial u}{\partial x_i},\theta\right)ds+\int_0^t(fu,\theta)ds.
\end{aligned}\tag{9.21}$$

Bound each term on the right side of (9.21) and use the coerciveness of $a(t,u,u)$. Then the equality (9.21) becomes

$$\frac{1}{2}|\theta(t)|^2+\frac{1}{2}|u(t)|^2\leq C\int_0^t\|h\|(\|u\|+|\theta|)ds+C\int_0^t(\|u\|^2+|\theta|^2)ds$$

where C is a constant independent of u and h. The Gronwall lemma applied in this last inequality gives the estimate (9.17).

■

Remark 9.5 *Let*

$$\begin{aligned}
&\tilde{a}_{ij}(x,t)=a_{ij}(x,T-t),\ \tilde{A}(t)=A(T-t)\\
&\tilde{b}_i(x,t)=-b_i(x,T-t),\ \tilde{c}(x,t)=-c(x,T-t)\\
&\tilde{d}_i(x,t)=d_i(x,T-t),\ \tilde{f}(x,t)=f(x,T-t),\ \tilde{h}(x,t)=h(x,T-t)
\end{aligned}\tag{9.22}$$

and

$$\begin{aligned}
&\tilde{u}(x,t)=u(x,T-t),\\
&\tilde{R}\tilde{u}=\tilde{u}''+\tilde{A}\tilde{u}+\tilde{b}_i\frac{\partial\tilde{u}'}{\partial x_i}+\tilde{c}\tilde{u}'+\tilde{d}_i\frac{\partial\tilde{u}}{\partial x_i}+\tilde{f}\tilde{u}.
\end{aligned}\tag{9.23}$$

Clearly if a_{ij}, b_i, c, d_i, f satisfy the hypotheses (9.6) then $\tilde{a_{ij}}$, $\tilde{b_i}$, $\tilde{c}$, $\tilde{d_i}$, $\tilde{f}$ satisfy the same hypotheses and reciprocally.

We introduce the problem

$$\left| \begin{array}{l} \tilde{R}\tilde{u} = \tilde{h} \text{ in } Q, \\ \tilde{u} = 0 \text{ in } \Sigma, \\ \tilde{u}(T) = u^0,\ \tilde{u}'(0) = -u^1 \text{ in } \Omega \end{array} \right. \tag{9.24}$$

and define in similar manner as in Problem (9.7) a weak solution $\tilde{u}$ of this problem. A direct computation shows that u is a weak solution of Problem (9.7) if and only if $\tilde{u}$ is a weak solution of Problem (9.24), u and $\tilde{u}$ related by (9.23). Thus we can prove a Theorem 9.3' and a Theorem 9.4' for the weak solution $\tilde{u}$ of (9.24) analogous to Theorem 9.3 and Theorem 9.4.

■

9.4 Inverse and Direct Inequality

The objective of this section it to obtain estimates for $\frac{\partial u}{\partial \nu}$, u the weak solution of the problem

$$\left| \begin{array}{l} L^* u = h \text{ in } Q, \\ u = 0 \text{ in } \Sigma, \\ u(0) = u^0,\ u'(0) = u^1 \text{ in } \Omega \end{array} \right. \tag{9.25}$$

where L^* the operator introduced in (9.2).

In the sequel of the section we will work with the operators L^* and L. We observe that, since the coefficients of L^* satisfy the condition (9.6), all the results of Section 9.3 remain true when one changes the operator R by L^*.

With the notations

$$a_{ij} = (\delta_{ij} - k'^2 x_i x_j) k^{-2},\ b_i = -2k' k^{-1} x_i \tag{9.26}$$

and after some computations, the operator L^* assumes the form

$$\begin{aligned} L^* u = u'' - \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j} \right) + \frac{1}{2}\frac{\partial}{\partial x_i}(b_i u') + \\ + \frac{1}{2}\frac{\partial}{\partial x_i}(b_i u)' + \frac{\partial}{\partial x_i}(n k'^2 k^{-2} x_i u). \end{aligned} \tag{9.27}$$

One has

$$a_0 \xi_i \xi_j \le a_{ij} \xi_i \xi_j \le a_1 \xi_i \xi_j,\ \forall \{x,t\}, \forall \xi \in \mathbb{R}^n,\ (a_0 > 0). \tag{9.28}$$

The energy of system (9.25) is

$$E(t) = \frac{1}{2}|u'(t)|^2 + \frac{1}{2} a(t, u(t), u(t)) \tag{9.29}$$

in particular

$$E_0 = E(0) = \frac{1}{2}|u^1|^2 + \frac{1}{2}a(0, u^0, u^0).$$

Theorem 9.5 *Let u be the weak solution of Problem (9.25). Then*
(i) if $h = 0$,

$$E_0 e^{-C_0} \leq E(t) \leq E_0 e^{C_0}, \ \forall t \in [0, \infty),$$

(ii) if $h \neq 0$,

$$E(t) \leq \left[2E_0 + \left(\int_0^T |h| dt\right)^2\right] e^{C_0}, \ \forall t \in [0, T],$$

where

$$C_0 = 2(1 + \tau k_1 M^2 + \tau^2 M^2 + n a_0 k_1^2)(a_0 k_0^3)^{-1}(l_1 + l_2) + \\ + 2(\lambda_1^{\frac{1}{2}} M + n)(n\tau + \tau + k_1)(a_0^{\frac{1}{2}} k_0^2 \lambda_1^{\frac{1}{2}})^{-1}(l_1 + l_2)$$

(see notations of Section 9.3 and 9.29).

Proof. We will prove the part (i). The second part will be obtained with the same arguments. Differentiating with respect to t the identity (iii) of Theorem 9.3, we find

$$E'(t) = \frac{1}{2}a'(t, u, u) + \frac{1}{2}\left(\frac{\partial b_i}{\partial x_i}u', u'\right) - (Pu, u')$$

or

$$E'(t) = -k'k^{-3}\left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\right) - (k'k''k - k'^3)k^{-3}\left|x_i \frac{\partial u}{\partial x_i}\right|^2 + \\ + nk'k^{-1}|u'|^2 - \left(d_i \frac{\partial u}{\partial x_i}, u'\right) - (fu, u'). \tag{9.30}$$

Recalling that

$$d_i = [(n+1)k'^2 - k''k]k^{-2}x_i, \quad f = [n(n+1)k'^2 - nk''k]k^{-2}$$

(see (9.2)), it follows from (9.30)

$$|E'(t)| \leq G(t)E(t)$$

where

$$G = 2(|k'| + M^2|k'k''k - k'^3| + n a_0 k^2 |k'|)(a_0 k^3)^{-1} + \\ + 2(\lambda_1^{\frac{1}{2}} M + n)|(n+1)k'^2 - k''k|(a_0^{\frac{1}{2}} \lambda_1^{\frac{1}{2}} k^2)^{-1}. \tag{9.31}$$

or

$$-G(t)E(t) \leq E'(t) \leq G(t)E(t). \tag{9.32}$$

Using the Hypotheses $(H3)$-$(H5)$, Section 9.2, on the function k we can bound each term that define G and (9.31) gives

$$\int_0^\infty G(t)dt \le C_0. \tag{9.33}$$

Combining (9.32) and (9.33) we conclude the proof of the theorem.

■

Next, one express an identity which will the fundamental to obtain estimates for $\frac{\partial u}{\partial \nu}$.

Theorem 9.6 *Leg $q = (q_l)$ be a vectorial field on $\overline{\Omega}$, $q \in [C^1(\overline{\Omega})]^n$. Then every weak solution u of Problem (9.25) verifies*

$$\begin{aligned}
&\frac{1}{2}\int_0^T\!\!\int_\Gamma a_{ij}\nu_i\nu_j q_l\nu_l\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt = \\
&= \left(u' + \frac{1}{2}\frac{\partial}{\partial x_i}[b_i u], q_l\frac{\partial u}{\partial x_l}\right)\Big|_0^T + \\
&+ \frac{1}{2}\int_0^T\!\!\int_\Omega \frac{\partial q_l}{\partial x_l}\left(u'^2 - a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\right)dxdt + \\
&+ (n+1)\int_0^T\!\!\int_\Omega \frac{k'^2}{k^2}x_i\frac{\partial u}{\partial x_i}q_l\frac{\partial u}{\partial x_l}dxdt + \\
&+ \int_0^T\!\!\int_\Omega a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial q_l}{\partial x_i}\frac{\partial u}{\partial x_l}dxdt + \frac{1}{2}\int_0^T\!\!\int_\Omega \frac{\partial b_i}{\partial x_i}\frac{\partial u}{\partial x_l}q_l u' dxdt + \\
&+ \frac{1}{2}\int_0^T\!\!\int_\Omega \frac{\partial b_i}{\partial x_l}\frac{\partial u}{\partial x_i}q_l u' dxdt + \frac{1}{2}\int_0^T\!\!\int_\Omega \frac{\partial b_i}{\partial x_i}z\frac{\partial q_l}{\partial x_l}u' dxdt + \\
&+ \frac{1}{2}\int_0^T\!\!\int_\Omega b_i\frac{\partial u}{\partial x_i}\frac{\partial q_l}{\partial x_l}u' dxdt - \frac{1}{2}\int_0^T\!\!\int_\Omega b_i u'\frac{\partial q_l}{\partial x_i}\frac{\partial u}{\partial x_l}dxdt - \\
&- \frac{1}{2}\int_0^T\!\!\int_\Omega n^2\frac{k'^2}{k^2}\frac{\partial q_l}{\partial x_l}u^2 dxdt - \int_0^T\!\!\int_\Omega h q_l\frac{\partial u}{\partial x_l}dxdt
\end{aligned} \tag{9.34}$$

Remark 9.6 *In order to facilitate the writing, we denote $\frac{\partial}{\partial x_i}$ by D_i and the product of the functions φ, ψ by $\varphi \cdot \psi$.*

Proof. First, one proves (9.34) for the solution u of Problem (9.25) with smooth data, that is, u given by Theorem 9.2. Then, the general case will follow by taking the limit in the identity with smooth solution. Thus $u(t) \in H^2(\Omega) \times H_0^1(\Omega)$, $u'(t) \in H_0^1(\Omega)$.

Multiply the equation $(9.25)_1$ by $q_l D_l u$. On each term of the product $L^*u \cdot q_l D_l u$ one uses the Green's formula or integrates by parts in t. It gives: For the first term

$$\int_0^T\!\!\int_\Omega u'' q_l D_l u dxdt = (u', q_l D_l u)\Big|_0^T + \frac{1}{2}\int_0^T\!\!\int_\Omega (D_l q_l)u'^2 dxdt \tag{9.35}$$

For the second term

$$
\begin{aligned}
&-(D_i[a_{ij}D_ju],q_lD_lu)=(a_{ij}D_ju,D_iq_lD_lu)+\\
&+(a_{ij}D_ju,q_lD_iD_lu)-\int_\Gamma a_{ij}D_ju\cdot q_lD_lu\cdot\nu_id\Gamma.
\end{aligned}
\tag{9.36}
$$

Applying the operator D_l on $a_{ij}D_jz\cdot q_lD_iz$ and noting that $a_{ij}=a_{ji}$, we deduce by using the Green's formula in the second integral on the right side of (9.36), that

$$
\begin{aligned}
&2(a_{ij}D_ju,q_lD_iD_lu)=-(D_la_{ij}\cdot D_ju\cdot q_l,D_iu)-\\
&-(a_{ij}D_ju\cdot D_lq_l,D_iu)+\int_\Gamma a_{ij}D_ju\cdot q_lD_iu\cdot\nu_ld\Gamma.
\end{aligned}
\tag{9.37}
$$

One has that $D_iu=\nu_i\frac{\partial u}{\partial\nu}$ on Γ (see J.L. Lions [39]), therefore

$$
\begin{aligned}
&\frac{1}{2}\int_\Gamma a_{ij}D_ju\cdot q_lD_iu\cdot\nu_ld\Gamma-\int_\Gamma a_{ij}D_ju\cdot q_lD_lu\cdot\nu_id\Gamma-\\
&-\frac{1}{2}\int_\Gamma a_{ij}\nu_i\nu_jq_l\nu_l\left(\frac{\partial u}{\partial\nu}\right)^2d\Gamma.
\end{aligned}
\tag{9.38}
$$

A direct computation on a_{ij} gives

$$
-\frac{1}{2}(D_la_{ij}\cdot D_jz\cdot q_l,D_iz)=\left(\frac{k'^2}{k}x_iD_iu,q_lD_lu\right)
\tag{9.39}
$$

Combining (9.35)-(9.37), we find the expression

$$
\begin{aligned}
&-(D_i[a_{ij}D_ju],q_lD_lu)=(a_{ij}D_ju,D_iq_lD_lu)+\\
&+\left(\frac{k'^2}{k^2}x_iD_iu,q_lD_lu\right)-\frac{1}{2}(a_{ij}D_ju\cdot D_lq_l,D_iz)-\\
&-\frac{1}{2}\int_\Gamma a_{ij}\nu_i\nu_jq_l\nu_l\left(\frac{\partial u}{\partial\nu}\right)^2d\Gamma
\end{aligned}
\tag{9.40}
$$

For the third term

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\int_\Omega D_i(b_iu')\cdot q_lD_ludxdt=-\frac{1}{2}\int_0^T\int_\Omega b_iu'D_iq_l\cdot D_ludxdt-\\
&-\frac{1}{2}\int_0^T\int_\Omega b_iu'q_lD_iD_ludxdt.
\end{aligned}
\tag{9.41}
$$

For the fourth term

$$
\begin{aligned}
&\frac{1}{2}\int_0^T\int_\Omega[D_i(b_iu)]'q_lD_ludxdt=\frac{1}{2}(D_i[b_iu],q_lD_lu)\Big|_0^T-\\
&-\frac{1}{2}\int_0^T\int_\Omega D_i(b_iu)\cdot q_lD_lu'dxdt,
\end{aligned}
\tag{9.42}
$$

and

$$
\begin{aligned}
&-\frac{1}{2}(D_i(b_i u), q_l D_l u') = \\
&= \frac{1}{2}([D_i b_i q_l D_l u + D_l b_i q_l D_i u + b_i q_l D_l D_i u], u') + \\
&+ \frac{1}{2}([u D_i b_i D_l q_l + b_i D_i u D_l q_l], u').
\end{aligned} \tag{9.43}
$$

(Note that $D_l D_i b_i = 0$.) From (9.42),(9.43) it follows that

$$
\begin{aligned}
&-\frac{1}{2}\int_0^T\int_\Omega [D_i(b_i u))]' q_l D_l u dx dt = \frac{1}{2}(D_i[b_i u], q_l D_l u)\Big|_0^T + \\
&+ \frac{1}{2}\int_0^T\int_\Omega D_i b_i \cdot q_l D_l u \cdot u' dx dt + \\
&+ \frac{1}{2}\int_0^T\int_\Omega D_l b_i q_l D_i u u' dx dt + \frac{1}{2}\int_0^T\int_\Omega b_i \cdot q_l D_l D_i u u' dx dt + \\
&+ \frac{1}{2}\int_0^T\int_\Omega u D_i b_i D_l q_l u' dx dt + \frac{1}{2}\int_0^T\int_\Omega b_i D_i u D_l q_l u' dx dt.
\end{aligned} \tag{9.44}
$$

If we add (9.41) and (9.44), one observes that the integrals involving $D_l D_i u$ are cancelled out.

For the fifth term

$$
\begin{aligned}
&\int_0^T\int_\Omega D_i\left(n\frac{k'^2}{k^2}x_i u\right)\cdot q_l D_l u dx dt = -\frac{1}{2}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2 D_l q_l dx dt + \\
&+ \int_0^T\int_\Omega n\frac{k'^2}{k^2}x_i D_i u \cdot q_l D_l u dx dt.
\end{aligned} \tag{9.45}
$$

Add (9.35), (9.40), (9.41), (9.44) and (9.45). As this addition is equals to $\int_0^T\int_\Omega h q_l D_l u dx dt$, we obtain the identity (9.34).

■

Let us consider again the notation of Section 9.2. The next inequality that we derive is named direct inequality for Problem (9.25).

Theorem 9.7 *Let z any weak solution of Problem (9.25). Then $\frac{\partial u}{\partial \nu} \in L^2(\Sigma)$ and*

$$
\int_0^T\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt \le C(T+1)\left[E_0 + \left(\int_0^T |h| dt\right)^2\right] + C_0 E_0^{\frac{1}{2}}\int_0^T |h| dt
$$

where C is a constant independent of u and T.

Proof. Theorem 9.5, part (ii), furnishes the estimate

$$
\left|\begin{aligned}
&|u'(t)|^2 + a(t, u(t), u(t)) \le C E_0 + C\left(\int_0^T |h| dt\right)^2, \\
&\forall t \in [0, T], \ (C = 4e^{C_0}).
\end{aligned}\right. \tag{9.46}
$$

Consider the identity of the Theorem 9.6 with a vector field q such that $q = \nu$ on Γ. One observes that, using the estimate (9.46), the integrals on Ω of (9.34) can be bounded by

$$C\left[E_0 + C\left(\int_0^T |h|dt\right)^2\right],$$

the integrals on Q by

$$C(T+1)\left[E_0 + C\left(\int_0^T |h|dt\right)^2\right],$$

and the integral $\int_0^T\int_\Omega hq_l\frac{\partial u}{\partial x_l}dxdt$ by

$$CE_0^{\frac{1}{2}}\int_0^T |h|dt + C\left(\int_0^T |h|dt\right)^2$$

On the other side

$$\frac{1}{2}\int_0^T\int_\Gamma a_{ij}\nu_i\nu_j\nu_l\nu_l\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt \geq \frac{1}{2}a_0\int_0^T\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt.$$

The above boundedness give the theorem.

■

Remark 9.7 *The Theorem 9.7 with $E(T)$ instead E_0 is also true for the weak solution u of the problem*

$$\left|\begin{array}{l} L^*u = h \text{ in } Q, \\ u = 0 \text{ in } \Sigma, \\ u(T) = u^0,\ u'(T) = u^1 \text{ in } \Omega. \end{array}\right.$$

For that we introduce the function $\tilde{k}(t) = k(T-t)$. With the coefficients a_{ij}, b_i, c, d_i, f of L^* we determine the coefficients $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}$, $\tilde{d}_i$, $\tilde{f}$ given by (9.22) of Remark 9.5. We then observe that the operator $\tilde{L}^*$ with coefficients $\tilde{a}_{ij}$, $\tilde{b}_i$, $\tilde{c}$, $\tilde{d}_i$, $\tilde{f}$ and the operator $\tilde{L}^*$ have the same form. The result then follows by applying Remark 9.5.

In order to show the inverse inequality one proves the following previous result:

Lemma 9.1 *Let $x^0 \in \mathbb{R}^n$. Then every solution u of Problem (9.25) with $h = 0$ verifies*

$$\left|\begin{array}{l} \frac{1}{2}\int_0^T\int_\Gamma a_{ij}\nu_i\nu_j\nu_l(x_l - x_l^0)\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt = \\ = \int_0^T E(t)dt + \left(u' + \frac{1}{2}\frac{\partial}{\partial x_i}[b_i u], (x_l - x_l^0)\frac{\partial u}{\partial x_l} + \frac{n-1}{2}u\right)\Big|_0^T + \\ + (n+1)\int_0^T\int_\Omega \frac{k'^2}{k^2}x_i\frac{\partial u}{\partial x_i}(x_l - x_l^0)\frac{\partial u}{\partial x_l}dxdt - \\ - (n+1)\int_0^T\int_\Omega \frac{k'}{k}\frac{\partial u}{\partial x_l}(x_l - x_l^0)u_l' dxdt + \\ + \frac{(n+1)}{4}\int_0^T\int_\Omega \frac{\partial b_i}{\partial x_i}uu'dxdt - \frac{(n+1)}{4}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2 dxdt. \end{array}\right. \tag{9.47}$$

Proof. Consider the identity of Theorem 9.6 with the particular vector field $q(x) = x - x^0$. Using the same arguments and notations of the proof of Theorem 9.6 and making the decompositions

- $\frac{n}{2}\int_0^T\int_\Omega (u'^2 - a_{ij}D_juD_iu)dxdt + \int_0^T\int_\Omega a_{ij}D_juD_iudxdt =$
  $= \frac{n-1}{2}\int_0^T\int_\Omega (u'^2 - a_{ij}D_juD_iu)dxdt + \int_0^T E(t)dt,$
- $\frac{n}{2}\int_0^T\int_\Omega D_ib_i\cdot uu'dxdt = \frac{n+1}{4}\int_0^T\int_\Omega D_ib_i uu'dxdt +$
  $+ \frac{n-1}{4}\int_0^T\int_\Omega D_ib_i\cdot uu'dxdt,$
- $\frac{n}{2}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt = \frac{n+1}{4}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt +$
  $+ \frac{n-1}{4}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt$

we obtain

$$\left|\begin{array}{l} \frac{1}{2}\int_0^T\int_\Gamma a_{ij}\nu_i\nu_j\nu_l q_l\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt = \left(u' + \frac{1}{2}D_i[b_iu], q_lD_lu\right)\Big|_0^T + \\ + \int_0^T E(t)dt + (n+1)\int_0^T\int_\Gamma \frac{k'^2}{k^2}x_iD_iu q_lD_ludxdt + \\ - (n+1)\int_0^T\int_\Omega \frac{k'}{k}D_luq_lu'dxdt + \frac{n+1}{4}\int_0^T\int_\Omega D_ib_iuu'dxdt - \\ - \frac{(n+1)}{4}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt + \\ + \frac{n-1}{2}\int_0^T\int_\Omega (u'^2 - a_{ij}D_juD_iu)dxdt + \\ + \frac{(n-1)}{2}\int_0^T\int_\Omega b_iD_iu\cdot u'dxdt + \frac{(n-1)}{4}\int_0^T\int_\Omega D_ib_iuu'dxdt - \\ - \frac{(n-1)}{4}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt. \end{array}\right. \tag{9.48}$$

Multiply the equation $L^*u = 0$ by u and integrate on Q. Use the Green's formula or the integration by parts in t on each term of the product $L^*u\cdot u$. This yields:

- $\int_0^T\int_\Omega u''udxdt = (u', u)\big|_0^T - \int_0^T\int_\Omega u'^2dxdt,$
- $\int_0^T \langle A(t)u, u\rangle dt = \int_0^T a(t, u, u)dt,$

- $\frac{1}{2}\int_0^T\int_\Omega D_i[b_iu]'\cdot udxdt = \frac{1}{2}(D_i[b_iu],u)\Big|_0^T -$
  $-\frac{1}{2}\int_0^T\int_\Omega D_ib_i\cdot uu'dxdt - \frac{1}{2}\int_0^T\int_\Omega b_iD_iu\cdot u'dxdt,$
- $\frac{1}{2}\int_0^T\int_\Omega D_i[b_iu']\cdot udxdt = -\frac{1}{2}\int_0^T\int_\Omega b_iu'D_iu,$
- $\int_0^T\int_\Omega D_i\left[n\frac{k'^2}{k^2}x_iu\right]\cdot udxdt = \frac{1}{2}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt.$

The addition on the last five equalities and the multiplication of the result by $\frac{(n-1)}{2}$ gives

$$\left|\begin{aligned}&\left(u'+\frac{1}{2}D_i[b_iu],\frac{n-1}{2}u\right)\Big|_0^T=\\&=\frac{(n-1)}{2}\int_0^T\int_\Gamma(u'^2-a_{ij}D_juD_iu)dxdt+\\&+\frac{(n-1)}{4}\int_0^T\int_\Gamma D_ib_iuu'dxdt+\frac{(n-1)}{2}\int_0^T\int_\Gamma b_iD_iu\cdot u'dxdt-\\&-\frac{(n-1)}{4}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt.\end{aligned}\right. \tag{9.49}$$

Using (9.49) in the last four terms on the right side of (9.48), we obtain the lemma.

■

In order to state the inverse inequality for the Problem (9.25) we introduce some notations. Theorem 9.5. part (i), says

$$C_1E_0 \le E(t) \le C_2E_0,\ \forall t\in[0,\infty)\ (C_1=e^{-C_0},\ C_2=e^{C_0}) \tag{9.50}$$

The time T_0 is defined by

$$T_0 = [2a_0^{\frac{1}{2}}R(x^0)+K_1+K_2+K_3]C_2C_1^{-1} \tag{9.51}$$

where

$$K_1 = \frac{2\tau[(n-1)M+2R(x^0)+2\lambda_1^{\frac{1}{2}}MR(x^0)]}{a_0k_0\lambda_1^{\frac{1}{2}}}$$
$$K_2 = \frac{2l_1(n+1)R(x^0)[\tau M+a_0^{\frac{1}{2}}k_0]}{a_0k_0^2}$$
$$K_3 = \frac{l_1n(n+1)[\tau M+a_0^{\frac{1}{2}}k_0]}{a_0k_0^2\lambda_1^{\frac{1}{2}}}$$

Remark 9.8 *We observe that if the function $k\equiv 1$ then $C_1=C_2=a_0=1$, $K_1=K_2=K_3=0$ that implies $T_0=2R(x^0)$. This is the time T_0 found in J.L. Lions [39] and in V. Komornik [26] for the equation $u''-\triangle u=0$.*

Theorem 9.8 *Let $T > T_0$. Then every weak solution u of Problem (9.25) with $h = 0$, verifies*

$$\frac{1}{2}R(x^0)a_1 \int_0^T\int_{\Gamma(x^0)} \left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt \geq C_1(T-T_0)E_0.$$

Proof The principal idea is to bound, using estimate (9.50), each integral on Q of the identity of Lemma 9.1 by an expression of the form

$$CE_0 \int_0^\infty |k'|dt.$$

We start bounding the first terms of the above identity. We have, making the same calculations as in [38] and [39],

$$\left|\left(u', (x_l - x_l^0)D_l u + \frac{n-1}{2}u\right)\right| \leq \frac{\mu}{2}|u'|^2 + \frac{1}{2\mu}R^2(x^0)a_0^{-1}a(t,u,u).$$

Making $\mu = \frac{R(x^0)}{a_0^{\frac{1}{2}}}$ and using the estimate (9.50) in the above inequality, we get

$$\left|\left(u', (x_l - x_l^0)D_l u + \frac{n-1}{2}u\right)\right| \leq R(x^0)a_0^{-\frac{1}{2}}C_2E_0.$$

that implies

$$\left(u', (x_l - x_l^0)D_l u + \frac{n-1}{2}u\right)\Big|_0^T \geq -R(x^0)a_0^{-\frac{1}{2}}C_2E_0. \tag{9.52}$$

Applying the Green's formula, we derive

$$\left(D_i[b_i u], (x_l - x_l^0)D_l u + \frac{n-1}{2}u\right) = \frac{n-1}{4}(u, b_iD_iu) +$$
$$+\frac{1}{2}\left(-\frac{2k'}{k}u + b_iD_iu, [x_l - x_l^0]D_l u\right)$$

and direct computations gives

- $\left|\frac{n-1}{4}(u, b_iD_iu)\right| \leq (n-1)\tau M C_2 \frac{E_0}{a_0k_0\lambda_1^{\frac{1}{2}}}$,

- $\left|\frac{1}{2}\left(-\frac{2k'}{k}u + b_iD_iu, [x_l - x_l^0]D_l u\right)\right| \leq 2\tau R(x^0)\frac{(1+\lambda_1^{\frac{1}{2}}M)C_2E_0}{a_0k_0\lambda_1^{\frac{1}{2}}}$

that implies

$$\begin{aligned}\left(D_i[b_i u], [x_l - x_l^0]D_l u + \frac{n-1}{2}u\right)\Big|_0^T &\geq \\ \geq -2\tau[(n-1)M + 2R(x^0) + 2\lambda_1^{\frac{1}{2}}MR(x^0)]\frac{C_2E_0}{a_0k_0\lambda_1^{\frac{1}{2}}}&.\end{aligned} \tag{9.53}$$

The integrals on Q of the identity (9.47) have the following bounds, after use of the estimate (9.50) and direct computation:

$$\begin{aligned} &\left|(n+1)\int_0^T\int_\Omega \frac{k'^2}{k^2}x_iD_iu(x_l-x_l^0)D_ludxdt\right| \leq \\ &\leq 2l_1(n+1)\tau MR(x^0)\frac{C_2E_0}{a_0k_0^2}, \end{aligned} \tag{9.54}$$

$$\begin{aligned} &\left|(n+1)\int_0^T\int_\Omega \frac{k'}{k}(x_l-x_l^0)D_lu\cdot u'dxdt\right| \leq \\ &\leq 2l_1(n+1)R(x^0)\frac{C_2E_0}{a_0^{\frac{1}{2}}k_0}, \end{aligned} \tag{9.55}$$

$$\left|\frac{(n+1)}{4}\int_0^T\int_\Omega D_ib_i\cdot uu'dxdt\right| \leq l_1n(n+1)\frac{C_2E_0}{a_0^{\frac{1}{2}}k_0\lambda_1^{\frac{1}{2}}}, \tag{9.56}$$

$$\left|\frac{(n+1)}{4}\int_0^T\int_\Omega n^2\frac{k'^2}{k^2}u^2dxdt\right| \leq l_1\tau n(n+1)M\frac{C_2E_0}{a_0k_0^2\lambda_1^{\frac{1}{2}}}. \tag{9.57}$$

Thus, using the estimates (9.52)-(9.57) in the identity (9.47), we obtain

$$\frac{1}{2}\int_0^T\int_\Gamma a_{ij}\nu_i\nu_j\nu_l(x_l-x_l^0)\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt \geq C_1(T-T_0)E_0. \tag{9.58}$$

The left side of (9.58) can be bounded as in J.L. Lions [39] by

$$\frac{1}{2}R(x^0)a_1\int_0^T\int_{\Gamma(x^0)}\left(\frac{\partial u}{\partial \nu}\right)^2 d\Gamma dt. \tag{9.59}$$

Combining (9.58) and (9.59) we finish the proof of the theorem.

■

9.5 Exact controlabillity

In this section we conclude the proof of Theorem 9.1. Let L the operator defined in (9.1), that is

$$Lu = u'' - \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + b_i\frac{\partial u'}{\partial x_j} + [(1-n)k'^2 - k''k]k^{-2}x_i\frac{\partial u}{\partial x_i}$$

where a_{ij}, b_i are defined in (9.26). Consider the problem

$$\left|\begin{aligned} &Lu = 0 \text{ in } Q, \\ &u = v \text{ in } \Sigma, \\ &u(0) = u^0,\ u'(0) = u^1 \text{ in } \Omega. \end{aligned}\right. \tag{9.60}$$

First of all, we define the concept of solution of Problem (9.60). Formal integration by parts on Q gives

$$\begin{aligned}\int_Q Lu\cdot zdxdt = &-\int_\Omega u'(0)z(0)dx + \int_\Omega u(0)z'(0)dx-\\ &-\int_\Omega b_i(0)\frac{\partial}{\partial x_i}u(0)z(0)dx + \int_\Sigma u\frac{\partial u}{\partial \nu_A}d\Gamma dt+\\ &+\int_Q whdxdt\end{aligned} \tag{9.61}$$

where z is the solution of the problem

$$\left|\begin{array}{l} L^*z = h \text{ in } Q,\\ z = 0 \text{ in } \Sigma,\\ z(T)=0,\ z'(T)=0 \text{ in } \Omega\end{array}\right. \tag{9.62}$$

and

$$\frac{\partial z}{\partial \nu_A} = a_{ij}(x,t)\frac{\partial z}{\partial x_j}\nu_i. \tag{9.63}$$

If $h \in L^1(0,T;L^2(\Omega))$, by Theorem 9.3 and Remark 9.5, we have that the solution z of Problem (9.62) verifies

$$\begin{aligned}&z\in C([0,T];H_0^1(\Omega))\cap C^1([0,T];L^2(\Omega))\\ &|z'(0)|+\|z(0)\|\le C\int_0^T |h|dt\end{aligned} \tag{9.64}$$

and by Theorem 9.7 and Remark 9.7,

$$\frac{\partial z}{\partial \nu}\in L^2(\Sigma),\quad \left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(\Sigma)} \le C\int_0^T |h|dt \tag{9.65}$$

with C a constant independent of z and h.

Motivated by (9.61)-(9.65) we introduce the following definition: let

$$u^0\in L^2(\Omega),\ u^1\in H^{-1}(\Omega),\ v\in L^2(\Sigma) \tag{9.66}$$

We say that $u\in L^\infty(0,T;L^2(\Omega))$ is a **solution defined by transposition** of Problem (9.60) with data u^0, u^1, v if

$$\begin{aligned}&\int_0^T (u,h)dt = \langle u^1, z(0)\rangle - (u^0, z'(0)) -\\ &-\left\langle \frac{2k'(0)}{k(0)}x_i\frac{\partial u^0}{\partial x_i}, z(0)\right\rangle - \int_0^T \left(v, \frac{\partial z}{\partial \nu_A}\right)_{L^2(\Gamma)} dt\end{aligned} \tag{9.67}$$

for every $h\in L^1(0,T;L^2(\Omega))$ where z is related to h by Problem (9.62).

Clearly the above solution u is unique. We also have from (9.64) and (9.65)

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} \leq C\left(|u^0| + \|u^1\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Sigma)}\right) \tag{9.68}$$

where C is a constant independent of u.

In order the prove the regularity of the solutions defined by transposition we introduce a previous result. Let $h \in \mathcal{D}(Q)$, $\mathcal{D}(Q)$ space of test function on Q, and z the weak solution of the problem

$$\left| \begin{array}{l} L^*z = h' \text{ in } Q, \\ z = 0 \text{ in } \Sigma, \\ z(0) = 0,\ z'(0) = 0 \text{ in } \Omega. \end{array} \right. \tag{9.69}$$

From Theorem 9.4 we have that z verifies the estimate

$$\|z(t)\| + |z'(t) - h(t)| \leq C\int_0^T \|h\|dt,\ \forall t \in [0,T], \tag{9.70}$$

where C is a constant independent of z and h. In virtue of Theorem 9.7 we obtain from (9.69) that $\frac{\partial z}{\partial \nu} \in L^2(\Sigma)$.

Lemma 9.2 *The solution z of (9.69) with $h \in \mathcal{D}(Q)$ verifies*

$$\left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(\Sigma)} \leq C\int_0^T \|h\|dt$$

where C is a constant independent of z and h.

Proof. We have by Theorem 9.6 that z verifies the identity (9.34). In what follows we bound each term of this identity by

$$C\left(\int_0^T \|h\|dt\right)^2,\ C \text{ constant independent of } z \text{ and } h. \tag{9.71}$$

We obtain by estimate (9.70) that

$$\left(z' + \frac{1}{2}\frac{\partial}{\partial x_i}[b_i z], q_l\frac{\partial z}{\partial x_l}\right)\Big|_0^T \text{ is bounded by (9.71).} \tag{9.72}$$

The equality ${z'}^2 = (z'-h)^2 + 2h(z'-h) + h^2$ gives

$$\begin{aligned} \frac{1}{2}\int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l}{z'}^2 dxdt &= \frac{1}{2}\int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l}(z'-h)^2 dxdt + \\ &+ \int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l}h(z'-h)dxdt + \frac{1}{2}\int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l}h^2 dxdt \end{aligned} \tag{9.73}$$

On the other hand, the last integral on the right side of (9.34) after integration by parts on Q becomes

$$\int_0^T\int_\Omega h' q_l \frac{\partial z}{\partial x_l} dxdt = \int_0^T\int_\Omega \frac{\partial h}{\partial x_l} q_l z' dxdt + \int_0^T\int_\Omega h \frac{\partial q_l}{\partial x_l} z' dxdt$$

Then the equality $z' = (z' - h) + h$ and Remark 9.4, part (i) applied in this last two integral furnish

$$\begin{aligned} -\int_0^T\int_\Omega h' q_l \frac{\partial z}{\partial x_l} dxdt &= -\int_0^T\int_\Omega \frac{\partial h}{\partial x_l}(z'-h)dxdt- \\ -\frac{1}{2}\int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l} h^2 dxdt &- \int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l} h(z'-h)dxdt \end{aligned} \tag{9.74}$$

The addition of (9.73) and (9.74) implies

$$\begin{aligned} &\int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l} z'^2 dxdt - \int_0^T\int_\Omega h' q_l \frac{\partial z}{\partial x_l} dxdt = \\ &= \frac{1}{2}\int_0^T\int_\Omega \frac{\partial q_l}{\partial x_l}(z'-h)^2 dxdt - \int_0^T\int_\Omega \frac{\partial h}{\partial x_l} q_l (z'-h)dxdt \end{aligned} \tag{9.75}$$

The estimate (9.70) applied on the right side of (9.75) permits to bound the left side of this equality by (9.71).

The other integrals on the right side of (9.34) can by bounded by (9.71) after use of the equality $z' = (z' - h) + h$ and estimate (9.70). Thus the identity (9.34), (9.72) and the last two boundedness give the lemma.

■

Theorem 9.9 *Every solution u defined by transposition of Problem (9.60) has the regularity*

$$u \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)) \tag{9.76}$$

and the linear map

$$\begin{array}{ccc} L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma) & \mapsto & C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)) \\ \{u^0, u^1, v\} & \mapsto & w \end{array}$$

is continuous.

Proof. First we prove that $u \in C([0,T]; L^2(\Omega))$. Fix u^0, u^1, v in the class (9.66). Let $(u^0_\mu), (u^1_\mu)$, (v_μ) be sequence of vectors of $H^1_0(\Omega)$, $L^2(\Omega)$ and $H^2_0(0,T;H^2(\Gamma))$, respectively, such that

$$u^0_\mu \to u^0 \text{ in } L^2(\Omega),\ u^1_\mu \to u^1 \text{ in } H^{-1}(\Omega), v_\mu \to v \text{ in } L^2(\Sigma). \tag{9.77}$$

Let $\tilde{v}_\mu$ be a function in $H_0^2(0,T;H^2(\Omega))$ such that $\gamma\tilde{v}_\mu = \{v_\mu, 0\}$, γ function trace on Γ, and y_μ the solution of the problem

$$\left| \begin{array}{l} Ly_\mu = -L\tilde{v}_\mu \text{ in } Q, \\ y_\mu = 0 \text{ on } \Sigma, \\ y_\mu(0) = {u_\mu}^0, \ y'_\mu(0) = {u_\mu}^1 \text{ in } \Omega. \end{array} \right.$$

Then by Theorem 9.3,

$$y_\mu \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)).$$

Then we have that $u_\mu = y_\mu + \tilde{v}_\mu$ is the solution defined transposition of Problem (9.60) with data u_μ^0, u_μ^1 and $u_\mu \in C([0,T];L^2(\Omega))$. Therefore, by (9.68),

$$\|u - u_\mu\|_{L^\infty(0,T;L^2(\Omega)} \leq C\left[|u^0 - u_\mu| + \|u^1 - u_\mu^1\|_{H^{-1}(\Omega)} + \|v - v_\mu\|_{L^2(\Sigma)}\right].$$

Taking the limit in this expression and using convergences (9.77) and the regularity of u_μ, we obtain that $u \in C([0,T];L^2(\Omega))$.

Now we consider $h \in \mathcal{D}(Q)$ and z the weak solution of the problem

$$\left| \begin{array}{l} L^*z = h' \text{ in } Q, \\ z = 0 \text{ in } \Sigma, \\ z(T) = 0, \ z'(T) = 0 \text{ in } \Omega. \end{array} \right. \tag{9.78}$$

Then by Theorem 9.4 and Remark 9.5 we have that

$$\|z(t)\| + |z'(t) - h(t)| \leq C\int_0^T \|h\|dt, \ \forall t \in [0,T], \tag{9.79}$$

and by Lemma 9.1

$$\left\|\frac{\partial z}{\partial \nu}\right\|_{L^2(\Sigma)} \leq C\int_0^T \|h\|dt. \tag{9.80}$$

The constants in (9.79) and (9.80) are independent of z and h.

We have that $u' \in H^{-1}(0,T;L^2(\Omega))$ because $u \in L^2(0,T;L^2(\Omega))$. Then

$$\langle u', h\rangle = -(u,h')_{L^2(Q)} = -\int_0^T (u,h')dt.$$

As u is a solution defined by transposition on Problem (9.60) one has from (9.78)

$$\int_0^T (u,h')dt = \langle u^1, z(0)\rangle - (u^0, z'(0)) - \\ - \left\langle \frac{2k'(0)}{k(0)} x_i \frac{\partial u^0}{\partial x_i}, z(0)\right\rangle - \int_0^T \left(v, \frac{\partial z}{\partial \nu_A}\right)_{L^2(\Gamma)} dt$$

From estimates (9.79) and (9.80) we then obtain

$$|\langle u', h\rangle| \leq C\left[|u^0| + \|u^1\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Sigma)}\right] \int_0^T \|h\| dt, \ \forall h \in \mathcal{D}(Q)$$

This implies by the density of $\mathcal{D}(Q)$ in $L^1(0,T;H_0^1(\Omega))$ that

$$u' \in L^\infty(0,T;H^{-1}(\Omega))$$

and

$$\|u'\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq C\left[|u^0| + \|u^1\|_{H^{-1}(\Omega)} + \|v\|_{L^2(\Sigma)}\right]. \tag{9.81}$$

By similar arguments used as in the first part of the proof and noting that $u'_\mu \in C([0,T];H^{-1}(\Omega))$ we conclude that $u' \in C([0,T];H^{-1}(\Omega))$.

The continuity of the linear application $\{u^0, u^1, v\} \mapsto u$ is obtained by (9.68) and (9.81).

■

Remark 9.9 *It is clear that we can also define the solution defined by transposition u of the backward problem*

$$\left|\begin{array}{l} Lu = 0 \ \textit{in } Q, \\ u = v \ \textit{on } \Sigma, \\ u(T) = u^0, \ u'(T) = u^1 \ \textit{in } \Omega \end{array}\right.$$

in a similar manner as in Problem (9.60) and Theorem 9.9 is also true for this solution u. This is a consequence of Remark 9.5 and 9.7.

Now we finish the proof of Theorem 9.1. Let φ be the weak solution of the problem

$$\left|\begin{array}{l} L^*\varphi = 0 \text{ in } Q, \\ \varphi = 0 \text{ in } \Sigma, \\ \varphi(0) = \varphi^0, \ \varphi'(0) = \varphi^1 \text{ in } \Omega \end{array}\right. \tag{9.82}$$

with $\{\varphi^0, \varphi^1\} \in H_0^1(\Omega) \times L^2(\Omega)$. Then by Theorems 9.7 and 9.8 one has

$$\varphi \in C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega)),$$

$\frac{\partial\varphi}{\partial\nu} \in L^2(\Sigma)$ and

$$C_1(T - T_0)E_0 \leq \left\|\frac{\partial\varphi}{\partial\nu}\right\|^2_{L^2(\Sigma(x^0))} \leq C_2(T+1)E_0 \tag{9.83}$$

where C_1, C_2 are constants independent of φ. With φ one constructs the solution defined by transposition ψ of the problem

$$\left| \begin{array}{l} L\psi = 0 \text{ in } Q, \\ \psi = \begin{cases} \dfrac{\partial \varphi}{\partial \nu} & \text{on } \Sigma(x^0), \\ 0 & \text{on } \Sigma \setminus \Sigma(x^0), \end{cases} \\ \psi(T) = 0,\ \psi'(T) = 0. \end{array} \right. \tag{9.84}$$

Then by Theorem 9.9 and Remark 9.9, ψ belongs to the class

$$\psi \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)).$$

We have

$$\left| \begin{array}{l} \langle L\psi, \varphi\rangle = -\left\langle \psi'(0) - \dfrac{2k'(0)}{k(0)} x_i \dfrac{\partial \psi(0)}{\partial x_i}, \varphi^0 \right\rangle + \\ + (\psi(0), \varphi^1) + \displaystyle\int_0^T \int_{\Gamma(y^0)} \frac{\partial \varphi}{\partial \nu} \frac{\partial \varphi}{\partial \nu_A} d\Gamma dt + \langle \psi, L^* \varphi \rangle \end{array} \right. \tag{9.85}$$

that implies

$$\left| \begin{array}{l} \left\langle \psi'(0) - \dfrac{2k'(0)}{k(0)} x_i \dfrac{\partial \psi(0)}{\partial x_i}, \varphi^0 \right\rangle - (\psi(0), \varphi^1) = \\ = \displaystyle\int_0^T \int_{\Gamma(x^0)} a_{ij} \frac{\partial \varphi}{\partial x_j} \nu_i \frac{\partial \varphi}{\partial \nu} d\Gamma dt. \end{array} \right. \tag{9.86}$$

The last expression induces the introduction of the following operator:

$$\begin{array}{rlcl} \Lambda: & H_0^1(\Omega) \times L^2(\Omega) & \to & H^{-1}(\Omega) \times L^2(\Sigma)) \\ & \{\varphi^0, \varphi^1\} & \mapsto & \Lambda\{\varphi^0, \varphi^1\} = \left\{ \psi'(0) - \frac{2k'(0)}{k(0)} x_i \frac{\partial \psi(0)}{\partial x_i}, -\psi(0) \right\}. \end{array}$$

By (9.86) and noting that

$$\frac{\partial \varphi}{\partial x_j} = \nu_j \frac{\partial \varphi}{\partial \nu_j} \text{ on } \Gamma,$$

we obtain

$$a_0 \left\| \frac{\partial \varphi}{\partial \nu} \right\|^2_{L^2(\Sigma(x^0))} \leq \langle \Lambda\{\varphi^0, \varphi^1\}, \{\varphi^0, \varphi^1\}\rangle \leq a_1 \left\| \frac{\partial \varphi}{\partial \nu} \right\|^2_{L^2(\Sigma(x^0))}.$$

This and (9.83) imply that Λ is injective. With $\{\tilde{\varphi}^0, \tilde{\varphi}^1\}$ one determine the weak solution $\tilde{\varphi}$ of Problem (9.82) and with $\frac{\partial \tilde{\varphi}}{\partial \nu}$, the solution defined by transposition $\tilde{\psi}$ of Problem (9.84). If we develop $\langle L\psi, \tilde{\varphi}\rangle$, one obtains as (9.85)

$$\langle \Lambda\{\varphi^0, \varphi^1\}, \{\tilde{\varphi}^0, \tilde{\varphi}^1\}\rangle = \int_0^T \int_{\Gamma(x^0)} a_{ij} \frac{\partial \tilde{\varphi}}{\partial x_j} \nu_i \frac{\partial \varphi}{\partial \nu} d\Gamma dt$$

and if we develop $\langle L\tilde{\psi}, \varphi\rangle$,

$$\langle \Lambda\{\tilde{\varphi}^0, \tilde{\varphi}^1\}, \{\varphi^0, \varphi^1\}\rangle = \int_0^T \int_{\Gamma(x^0)} a_{ij} \frac{\partial \varphi}{\partial x_j} \nu_i \frac{\partial \tilde{\varphi}}{\partial \nu} d\Gamma dt.$$

Observing that

$$\frac{\partial \varphi}{\partial x_j} = \nu_j \frac{\partial \varphi}{\partial \nu} \text{ on } \Gamma,$$

we then obtain from the last two equalities that Λ is self-adjoint. Thus

$$\Lambda \text{ is an isomorphism from } H_0^1(\Omega) \times L^2(\Omega) \text{ onto } H^{-1}(\Omega) \times L^2(\Omega) \tag{9.87}$$

Let $\{u^0, u^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$. Then by (9.87), there exists $\{\varphi^0, \varphi^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\Lambda\{\varphi^0, \varphi^1\} = \left\{ u^1 - \frac{2k'(0)}{k(0)} x_i \frac{\partial u^0}{\partial x_i}, -u^0 \right\}.$$

With $\{\varphi^0, \varphi^1\}$ one determines the weak solution φ of Problem (9.82) and with $\frac{\partial \varphi}{\partial \nu}$, the solution defined by transposition ψ of Problem (9.84). Then we have that $u = \psi$ satisfies all the required conditions of Theorem 9.1.

The expression (9.85) is justified by approximation's arguments that hold for smooth solutions φ and ψ. Analogously for the other expressions. Thus we have concluded the proof of Theorem 9.1.

■

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Chapter 10

Exact Controllability for the Wave Equation in Domains with Variable Boundary

10.1 Introduction.

In this chapter we are interested in the exact boundary controllability of the system:

$$\left| \begin{array}{l} \widehat{u}'' - \triangle \widehat{u} = 0 \text{ in } \widehat{Q}, \\ \widehat{u} = \widehat{v} \text{ in } \widehat{\Sigma}, \\ \widehat{u}(0) = \widehat{u}^0, \ \widehat{u}'(0) = \widehat{u}^1 \text{ in } \Omega_0 \end{array} \right. \tag{*}$$

where $\widehat{Q}$ is a non cylindrical domain of $\mathbb{R}^{n+1}$. The result is obtained by transforming the problem $\widehat{Q}$ in a problem defined in a cylindrical domain Q and the showing that these two problems are equivalent. The result in Q is studied by applying the HUM of J.L.Lions. [1]

Let Ω be an open boundary set of $\mathbb{R}^n$ with boundary Γ of class C^2, which, without loss of generality, can be assumed containing the origin of $\mathbb{R}^n$, and $k : [0, \infty[\rightarrow [0, \infty[$ a continuously differentiable function. Let us consider the subsets Ω_t of $\mathbb{R}^n$ given by

$$\Omega_t = \{x \in \mathbb{R}^n ; x = k(t)y, y \in \Omega\}, \ 0 \leq t \leq T < \infty,$$

whose boundaries are denoted by Γ_t, and $\widehat{Q}$ the non cylindrical domain of $\mathbb{R}^{n+1}$,

$$\widehat{Q} = \bigcup_{0<t<T} \Omega_t \times \{t\} \tag{10.1}$$

[1]This part is a paper that was published for one of Authors in Revista Matemática, Universidad Complutense de Madrid, 9 (1996), pp. 435-457.

with lateral boundary

$$\widehat{\Sigma} = \bigcup_{0<t<T} \Gamma_t \times \{t\}.$$

Graphically it wold be We have the following system:

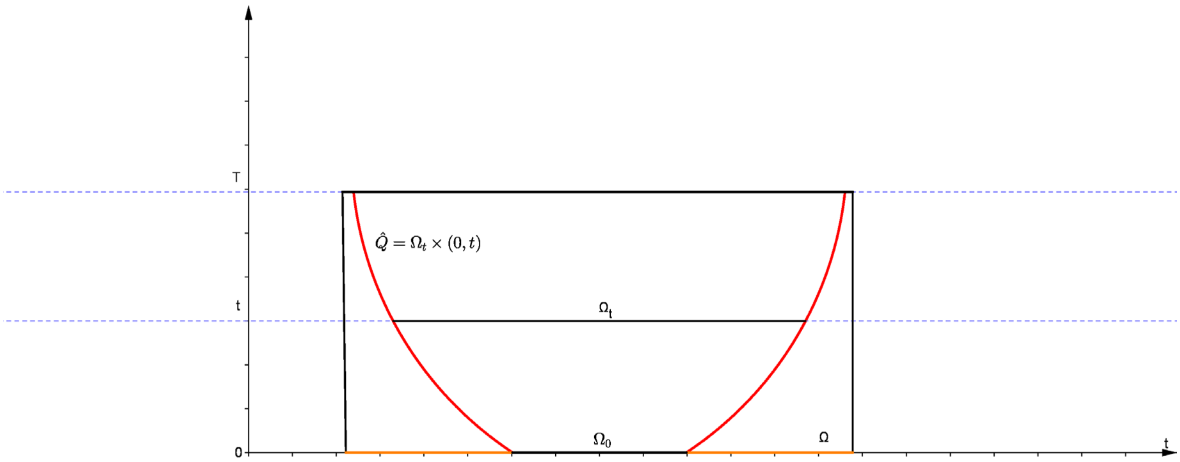

Figure 10.1: Non-cylindrical Domain

$$\left| \begin{array}{l} \widehat{u}'' - \Delta \widehat{u} = 0 \quad \text{in} \quad \widehat{Q}, \\ \widehat{u} = \widehat{v} \quad \text{on} \quad \widehat{\Sigma}, \\ \widehat{u}(0) = \widehat{u}^0, \widehat{u}' = \widehat{u}^1 \quad \text{on} \quad \Omega_0 \end{array} \right. \qquad (*)$$

where $\widehat{u}''$ stands for $\frac{\partial^2 \widehat{u}}{\partial t^2}$ and $\widehat{u}(0)$, $\widehat{u}'(0)$ denote, respectively, the functions $x \mapsto \widehat{u}(x,0)$, $x \mapsto \widehat{u}'(x,0)$. Here $\widehat{v}$ is the control variable, that is, it acts on the system $(*)$ through the lateral boundary $\widehat{\Sigma}$.

The problem of exact controllability for system $(*)$ states as follows: Given $T > 0$ large enough, is it possible, for every initial data $\{\widehat{u}^0, \widehat{u}^1\}$ in an appropriate space to a find a control $\widehat{v}$ driving the system to rest at time T, i.e., such that the solution $\widehat{u}(x,t)$ of $(*)$ satisfies

$$\widehat{u}(T) = 0, \;\; \widehat{u}'(T) = 0? \qquad (10.2)$$

We show that system $(*)$ is exactly controllable. Our approach consists first in transforming $(*)$, by using $k(t)$, in a system defined in the cylindrical domain $Q = \Omega \times]0,T[$. This system will have the following form:

$$\left| \begin{array}{l} u'' - \dfrac{\partial}{\partial y_i}\left(a_{ij}(y,t)\dfrac{\partial u}{\partial y_i}\right) + b_i(y,t)\dfrac{\partial u'}{\partial y_i} + d_i(y,t)\dfrac{\partial u}{\partial y_i} = 0 \text{ in } Q, \\ u = v \text{ in } \Sigma = \Gamma \times]0,T[, \\ u(0) = u^0, \;\; u'(0) = u^1 \text{ in } \Omega. \end{array} \right. \qquad (**)$$

Remark 10.1 *Here and in what follows the summation convention of repeated indices is adopted.*

Then we show that the study of the exact controllability problem for $(*)$ reduces to the study of the controllability for system $(**)$. The control $\widehat{v}$ will be expressed in function of a weak solution $\widehat{\theta}$ of the wave equation in the non cylindrical domain $\widehat{Q}$. For that, an appropriate change of variables is needed.

The controllability for system $(**)$ was analysed in the Chapter 9. The Hilbert Uniqueness Method (HUM) of J.L.Lions [39] is used in this analysis.

The existence of solutions of the initial boundary value problem for the nonlinear wave equation in general non cylindrical domain $\widehat{Q}$ was studied among other authors by J.L.Lions [42], L.A.Medeiros [46], when $\widehat{Q}$ is increasing and by C. Bardos and J. Cooper [3] when $\widehat{Q}$ is time like. A. Inoue [24] also analised this type of problems. The linear case was treated by J. Sikorav [62] when $\widehat{Q}$ is time like. He used tools of Differential Topology. The exact internal controllability problem for the wave equation in non cylindrical domains was treated by C. Bardos and G. Cheng [2]. They did not use HUM.

Remark 10.2 *The non cylindrical domain $\widehat{Q}$ that we have considered in $(*)$ is time like but it is not necessarily increasing or decreasing. This occurs because the derivative $k'(t)$ does not have sign condition. $\widehat{Q}$ is named time like when the unit normal vector $\eta = (\eta_x, \eta_t)$ to $\widehat{\Sigma}$, directed towards the exterior of $\widehat{Q}$, satisfies $|\eta_t| < |\eta_x|$.*

The Plan of this Chapter is organized as follows.

- Main result.
- Summary of Results on the Cylinder.
- Spaces on the Non Cylindrical Domain.
- Proof of the Main Result.

10.2 Main Result

Let us introduce some notations (cf. J.L. Lions [39]). Let $y^0 \in \mathbb{R}^n$, $m(y) = y - y^0$ and $\nu(y)$ the unit normal vector at $y \in \Gamma$, directed towards the exterior of Ω. Consider the sets

$$\Gamma(y^0) = \{y \in \Gamma; m(y) \cdot \nu(y) \geq 0\}, \ \Sigma(y^0) = \Gamma(y^0) \times]0, T[$$

and the corresponding sets in the (x,t)-coordinates,

$$\Gamma_t(y^0) = \{x \in \Gamma_t; x = k(t)y,\ y \in \Gamma(y^0)\},\ 0 \le t \le T$$

and

$$\widehat{\Sigma}(y^0) = \bigcup_{0<t<T} \Gamma_t(y^0) \times \{t\}$$

In the definition of $\Gamma(y^0)$, $\cdot$ denotes the scalar product in $\mathbb{R}^n$. We represent by $\eta = (\eta_x, \eta_t)$ the unit normal vector to $\widehat{\Sigma}$, directed towards the exterior of $\widehat{Q}$ and by ν^* the vector $\frac{\eta_x}{|\eta_x|}$.
Let

$$R(y^0) = \sup_{y\in\Omega} |m(y)|, \quad M = \sup_{y\in\Omega} |y|$$

and λ_1 the first eigenvalue of the spectral problem $-\triangle\varphi = \lambda\varphi$, $\varphi \in H_0^1(\Omega)$.

We make the following assumptions:

$$\text{The boundary } \Gamma \text{ of } \Omega \text{ is } C^2 \tag{H1}$$

and concerning the function k,

$$k \in W^{3,\infty}_{loc}(]0,\infty[) \tag{H2}$$

$$0 < k_0 = \inf_{t\ge0} k(t),\ \sup_{t\ge0} k(t) = k_1 < \infty \tag{H3}$$

$$\sup_{t\ge0} |k'(t)| = \tau < \frac{1}{M} \tag{H4}$$

$$l_1 = \int_0^\infty |k'(t)|dt < \infty,\ l_2 = \int_0^\infty |k''(t)|dt < \infty. \tag{H5}$$

Hypothesis (H4) implies that the non cylindrical domain $\widehat{Q}$ is time like. The unit outer normal vector $\eta(x,t)$ to $\widehat{\Sigma}$ is given in Remark 10.5.

All the scalar functions considered in the chapter will be real-valued.
In $\widehat{Q}$, $\widehat{Q}$ defined by (10.1), we have the following system:

$$\left|\begin{array}{l} \widehat{u}'' - \triangle\widehat{u} = 0 \text{ in } \widehat{Q}, \\ \widehat{u} = \begin{cases} \widehat{v} & \text{on } \widehat{\Sigma}(y^0), \\ 0 & \text{on } \widehat{\Sigma}\setminus\widehat{\Sigma}(y^0), \end{cases} \\ \widehat{u}(0) = \widehat{u}^0,\ \widehat{u}'(0) = \widehat{u}^1 \text{ in } \Omega_0. \end{array}\right. \tag{10.3}$$

In (10.12) we will give an explicit value for the minimal controllability time T_0 depending on n, $R(y^0)$, λ_1, the function k and on the geometry of Ω, and in (10.23), an isomorphism

$$\Lambda_1 : L^2(\Omega_0) \times H^{-1}(\Omega_0) \mapsto H_0^1(\Omega_0) \times L^2(\Omega_0),\ \Lambda_1\{\widehat{u}^0, \widehat{u}^1\} = \{\widehat{\theta}^0, \widehat{\theta}^1\}$$

which allows to compute the control $\widehat{v}$ for the initial data $\{\widehat{u}^0, \widehat{u}^1\}$.

Now we state the main result of the problem.

Theorem 10.1 *We assume that hypotheses $(H1)$-$(H5)$ are satisfied. Let $T > T_0$. Then, for each initial data $\{\widehat{u}^0, \widehat{u}^1\}$ belonging to $L^2(\Omega_0) \times H^{-1}(\Omega_0)$, there exists a control $\widehat{v} \in L^2(0, T; L^2(\Gamma_t(y^0)))$ such that the solution $\widehat{u}$ of system (10.3) satisfies the final condition (10.2). Moreover, the control $\widehat{v}$ has the form $\widehat{v} = \frac{\partial\widehat{\theta}}{\partial\nu^*}$ where $\widehat{\theta}$ is the weak solution of the problem*

$$\left| \begin{array}{l} \widehat{\theta}'' - \triangle\widehat{\theta} = 0 \ \ in \ \widehat{Q}, \\ \widehat{\theta} = 0 \ \ in \ \widehat{\Sigma}, \\ \widehat{\theta}(0) = \widehat{\theta}^0, \ \widehat{\theta}'(0) = \widehat{\theta}^1 \ \ in \ \Omega_0 \end{array} \right.$$

with $\Lambda_1\{\widehat{u}^0, \widehat{u}^1\} = \{\widehat{\theta}^0, \widehat{\theta}^1\}$.

The next three sections will be devoted to the proof of the above theorem.

10.3 Summary of Results on the Cylinder

Is this section we list the results on the cylinder Q that we will use in Section 5. Its proof can be found in chapter 9.

We consider the operator

$$Lw = w'' - \frac{\partial}{\partial y_i}\left(a_{ij}(y,t)\frac{\partial w}{\partial y_j}\right) + b_i(y,t)\frac{\partial w'}{\partial y_i} + d_i(y,t)\frac{\partial w}{\partial y_i} \tag{10.4}$$

where

$$a_{ij}(y,t) = (\delta_{ij} - k'^2 y_i y_j)k^{-2},$$

$$b_i(y,t) = -2k'k^{-1}y_i, \quad d_i(y,t) = [(1-n)k'^2 - k''k]k^{-2}y_i.$$

Then for z test function on Q, we have

$$\int_0^T\!\!\int_\Omega (Lw)zdydt = \int_0^T\!\!\int_\Omega w\left[z'' - \frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial z}{\partial y_j}\right) + \frac{\partial}{\partial y_i}(b_i z)' - \frac{\partial}{\partial y_i}(d_i z)\right]dydt =$$
$$= \int_0^T\!\!\int_\Omega wL^*zdydt.$$

We obtain

$$\frac{\partial}{\partial y_i}(b_i z)' = b_i\frac{\partial z'}{\partial y_i} - 2nk'k^{-1}z' + (2k'^2 - 2k''k)k^{-2}y_i\frac{\partial z}{\partial y_i} +$$
$$+ (2nk'^2 - 2nk''k)k^{-2}z - \frac{\partial}{\partial y_i}(d_i z) = [k''k - (1-n)k'^2]R^{-2}y_i\frac{\partial z}{\partial y_i} +$$
$$+ [nk''k - n(1-n)k'^2]k^{-2}z.$$

Thus L^*z, the formal adjoint of L, has the form

$$L^*z = z'' - \frac{\partial}{\partial y_i}\left(a_{ij}(y,t)\frac{\partial z}{\partial y_j}\right) + b_i(y,t)\frac{\partial z'}{\partial y_i} + Pz \tag{10.5}$$

where

$$Pz = -2nk'k^{-1}z' + [(n+1)k'^2 - k''k]k^{-2}y_i\frac{\partial z}{\partial y_i}+$$
$$+ [n(n+1)k'^2 - nk''k]R^{-2}z.$$

Let us consider the problem

$$\left|\begin{array}{l} L^*z = h \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \\ z(0) = z^0,\ z'(0) = z^1 \text{ in } \Omega \end{array}\right. \tag{10.6}$$

with data

$$z^0 \in H_0^1(\Omega),\ z^1 \in L^2(\Omega),\ h \in L^1(0,T;L^2(\Omega)). \tag{10.7}$$

A function $z : Q \to \mathbb{R}$ will be called a **weak solution** of Problem (10.6) if z belongs to the class

$$z \in L^\infty(0,T;\in H_0^1(\Omega)),\quad z' \in L^\infty(0,T;L^2(\Omega)),$$

satisfies the equation

$$-\int_0^t (z',\xi')dt + \int_0^T a(t,z,\xi)dt + \int_0^T \left\langle b_i\frac{\partial z'}{\partial y_i},\xi\right\rangle dt+$$
$$+\int_0^T (Pz,\xi)dt = \int_0^T (h,\xi)dt,$$
$$\forall \xi \in L^2(0,T;H_0^1(\Omega)),\ \xi' \in L^2(0,T;L^2(\Omega)),\ \xi(0) = \xi(T) = 0$$

and the initial conditions

$$z(0) = z^0,\quad z'(0) = z^1.$$

Here $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$, $\langle\cdot,\cdot\rangle$ the duality pairing between F' and F, F being a generic space and F' its dual, and

$$a(t,z,\xi) = \int_\Omega a_{ij}(y,t)\frac{\partial z}{\partial y_j}\frac{\partial \xi}{\partial y_i}dy.$$

We observe that if z is a weak solution of Problem (10.6) then z' is weakly continuous from $[0,T]$ with values in $L^2(\Omega)$. Therefore the above initial condition $z'(0)$ makes sense. The regularity of z' follows from $z' \in L^\infty(0,\infty;L^2(\Omega))$ and $z'' \in L^\infty(0,\infty;H^{-1}(\Omega))$.

Concerning to Problem (10.6) we have the following result:

Theorem 10.2 *For each data z^0, z^1, h in the class (10.7), there exists a unique weak solution z of Problem (10.6). This solution has the regularity:*

$$z \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$$

and

$$\frac{\partial z}{\partial \nu} \in L^2(0,T; L^2(\Gamma)). \tag{10.8}$$

From (10.8) it follows that $\frac{\partial z}{\partial \nu_A}$ belongs to $L^2(0,T;L^2(\Gamma))$ where

$$\frac{\partial z}{\partial \nu_A} = a_{ij}(y,t)\frac{\partial z}{\partial y_j}\nu_i.$$

We obtain all the above results it instead of Problem (10.6) we consider the backward problem

$$\left| \begin{array}{l} L^* z = h \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \\ z(T) = z^0, \ z'(T) = z^1 \text{ in } \Omega. \end{array} \right. \tag{10.9}$$

Let us consider the problem

$$\left| \begin{array}{l} Lu = 0 \text{ in } Q, \\ u = g \text{ on } \Sigma, \\ u(0) = u^0, \ u'(0) = u^1 \text{ in } \Omega \end{array} \right. \tag{10.10}$$

with data

$$u^0 \in L^2(\Omega), \ u^1 \in H^{-1}(\Omega), \quad g \in L^2(0,T;L^2(\Gamma)). \tag{10.11}$$

We say that $u \in L^\infty(0,T;L^2(\Omega))$ is a **solution defined by transposition** of Problem (10.10) if

$$\int_0^T (u,h)dt = \langle u^1, z(0)\rangle - (u^0, z'(0)) - \left\langle \frac{2k'(0)}{k(0)} y_i \frac{\partial u^0}{\partial y_i}, z(0) \right\rangle - \\ - \int_0^T \left(g, \frac{\partial z}{\partial \nu_A} \right)_{L^2(\Gamma)} dt$$

for every $h \in L^1(0,T;L^2(\Omega))$ where z is related to h by Problem (10.9) with $z^0 = z^1 = 0$.

We have the following result:

Theorem 10.3 *For each data u^0, u^1, g in the class (10.11), there exists a unique solution defined by transposition w of Problem (10.10). This solution has the regularity*

$$u \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega)).$$

We can change the initial data at time $t = 0$ by final data at time $t = T$ in Problem (10.10) and obtain the same above result.

In the sequel we introduce some constants in order to state the main result of this chapter. By hypotheses $(H3)$,$(H4)$ of Section 10.2 one has that there exists a unique positive constant α_0 such that

$$a_{ij}(y,t)\xi_i\xi_j \geq \alpha_0\xi_i\xi_j, \quad \forall\{y,t\} \in \Omega \times [0,\infty), \ \forall\xi \in \mathbb{R}^n.$$

With this and the notations of Section 10.2, we define:

$$\begin{aligned} &C_0 = 2(1 + \tau k_1 M^2 + \tau^2 M^2 + n\alpha_0 k_1^2)(\alpha_0 k_0^3)^{-1}(l_1 + l_2) \\ &+ 2(\lambda_1^{\frac{1}{2}} M + n)(n\tau + \tau + k_1)(\alpha_0^{\frac{1}{2}} k_0^2 \lambda_1^{\frac{1}{2}})^{-1}(l_1 + l_2), \\ &C_1 = e^{-C_0}, \quad C_2 = e^{C_0}. \end{aligned}$$

The minimal controllability time T_0 is then defined by

$$T_0 = [2\alpha_0^{-\frac{1}{2}} R(y^0) + K_1 + K_2 + K_3]C_2 C_1^{-1} \tag{10.12}$$

where

$$\begin{aligned} K_1 &= \frac{2\tau[(n-1)M + 2R(y^0) + 2\lambda_1^{\frac{1}{2}} M R(y^0)]}{\alpha_0 k_0 \lambda_1^{\frac{1}{2}}} \\ K_2 &= \frac{2l_1(n+1)R(y^0)[\tau M + \alpha_0^{\frac{1}{2}} k_0]}{\alpha_0 k_0^2} \\ K_3 &= \frac{l_1 n(n+1)[\tau M + \alpha_0^{\frac{1}{2}} k_0]}{\alpha_0 k_0^2 \lambda_1^{\frac{1}{2}}} \end{aligned}$$

We consider the problem

$$\left| \begin{array}{l} Lu = 0 \text{ in } Q, \\ u = \begin{cases} g \text{ on } \Sigma(y^0), \\ 0 \text{ on } \Sigma \setminus \Sigma(y^0), \end{cases} \\ u(0) = u^0, \ u'(0) = u^1 \text{ in } \Omega. \end{array} \right. \tag{10.13}$$

We have the following exact controllability result:

Theorem 10.4 *Let $T > T_0$, T_0 given by (10.12). Then for every $\{u^0, u^1\} \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control $g \in L^2(\sum(y^0))$ such that the solution defined by transposition u of Problem (10.13) satisfies*

$$u(T) = 0, \quad u'(T) = 0.$$

Remark 10.3 *We observe that if $k(t) \equiv 1$ then $K_1 = K_2 = K_3 = 0$, $C_1 = C_2 = 1$ and $\alpha_0 = 1$. Therefore $T_0 = 2R(y^0)$. Thus, in this case T_0 coincides with the minimal controllability time obtained earlier by J.L. Lions [39] and V. Komornik [26] for the wave equation $u'' - \triangle u = 0$.*

Let φ be the weak solution of problem

$$\left| \begin{array}{l} L^*\varphi = 0 \text{ in } Q, \\ \varphi = 0 \text{on } \Sigma, \\ \varphi(0) = \varphi^0, \ \varphi'(0) = \varphi^1 \text{ in } \Omega \end{array} \right. \tag{10.14}$$

with $\{\varphi^0, \varphi^1\} \in H_0^1(\Omega) \times L^2(\Omega)$, and ψ the solution defined by transposition of problem

$$\left| \begin{array}{l} L\psi = 0 \text{ in } Q, \\ \psi = \begin{cases} \frac{\partial \varphi}{\partial \nu} \text{ on } \Sigma(y^0), \\ 0 \text{ on } \Sigma \setminus \Sigma(y^0), \end{cases} \\ \psi(T) = 0, \ \psi'(0) = 0 \text{ in } \Omega. \end{array} \right. \tag{10.15}$$

With these last two problems, we introduce the operator Λ,

$$\begin{array}{ccc} H_0^1(\Omega) \times L^2(\Omega) & \to & H^{-1}(\Omega) \times L^2(\Omega) \\ \{\varphi^0, \varphi^1\} & \mapsto & \Lambda\{\varphi^0, \varphi^1\} = \left\{ \psi'(0) - \frac{2k'(0)}{k(0)} y_i \frac{\partial \psi(0)}{\partial y_i}, -\psi(0) \right\} \end{array} \tag{10.16}$$

The proof of Theorem 10.4 is reduced to prove that the operator

$$\Lambda \text{ is an isomorphism from } H_0^1(\Omega) \times L^2(\Omega) \text{ onto } H^{-1}(\Omega) \times L^2(\Omega).$$

This is done by showing, by multiplier techniques, that the following observability inequality holds for $T > T_0$:

$$\frac{1}{2}|\varphi^1|^2 + \frac{1}{2}a(0; \varphi^0, \varphi^0) \leq C \int_0^T \int_{\Gamma(y^0)} \left| \frac{\partial \varphi}{\partial \nu} \right|^2 d\Gamma dt$$

where φ is the solution of problem (10.14). We refer to Chapter 9 for the technical details.

Remark 10.4 *In system (10.15) we can consider $\frac{\partial \varphi}{\partial \nu_A}$ instead $\frac{\partial \varphi}{\partial \nu}$ and to obtain also the exact controllability for system (10.13). On the other side if $\varphi(y,t) = k^n(t)\theta(k(t)y, t)$, $x = k(t)y$, then*

$$\begin{aligned} \frac{\partial \varphi}{\partial \nu_A}(y,t) &= \left(\delta_{ij} - k'^2 y_i y_j \right) k^{-2} \frac{\partial \varphi}{\partial y_j}(y,t)\nu_i(y) = \\ &= \left(\delta_{ij} - k^{-2} k'^2 x_i x_j \right) k^{n-1} \frac{\partial \theta}{\partial x_j}(x,t) \nu_i^*(x,t). \end{aligned} \tag{10.17}$$

and

$$\frac{\partial \varphi}{\partial \nu}(y,t) = k^{n+1}\frac{\partial \theta}{\partial \nu^*}(x,t).$$

(For the computations see (10.35). We note that the second member of (10.17) is not a known derivative of the function θ. For this reason we consider $\frac{\partial\varphi}{\partial\nu}$ instead $\frac{\partial\varphi}{\partial\nu_A}$ in (10.15).

10.4 Spaces on the Non Cylindrical Domain

Let $\widehat{u} : \widehat{Q} \to \mathbb{R}$ be a function such that

$$\widehat{u}(x,t) = k^{-n}(t)\xi\left(\frac{x}{k(t)}, t\right), \ \xi \in L^p(0,T; W_0^{m,q}(\Omega)) \tag{10.18}$$

Then we have $\widehat{u}(t) \in W_0^{m,q}(\Omega_t)$ $a.e$ t in $]0,T[$ and

$$\|\widehat{u}(t)\|_{W_0^{m,q}(\Omega_t)} = k^{\frac{n}{q}-m-n}(t)\|\xi(t)\|_{W_0^{m,q}(\Omega)}.$$

Therefore,

$$C_3\|\xi(t)\|_{W_0^{m,q}(\Omega)} \le \|\widehat{u}(t)\|_{W_0^{m,q}(\Omega_t)} \le C_4\|\xi(t)\|_{W_0^{m,q}(\Omega)} \tag{10.19}$$

Here and in what follows C_3, C_4 will denote generic positive constants which are independent of $\widehat{u}$ and ξ.

We denote by $L^p(0,T; W_0^{m,q}(\Omega_t))$ $(1 \le p \le \infty,\ 1 \le q < \infty$, m a non-negative integer) the space of (class of) function $\widehat{u} : \widehat{Q} \to \mathbb{R}$ such that there exists $\xi \in L^p(0,T; W_0^{m,q}(\Omega))$ verifying (10.18), equipped with the norm

$$\|\widehat{u}\|_{L^p(0,T;W_0^{m,q}(\Omega_t))} = \left(\int_0^T \|\widehat{u}(t)\|^p_{W_0^{m,q}(\Omega_t)} dt\right)^{\frac{1}{p}},\ 1 \le p < \infty$$

$$\|\widehat{u}\|_{L^\infty(0,T;W_0^{m,q}(\Omega_t))} = \operatorname*{ess\,sup}_{t\in]0,T[} \|\widehat{u}(t)\|_{W_0^{m,q}(\Omega_t)}.$$

By (10.19), the space $X = L^p(0,T; W_0^{m,q}(\Omega_t))$ is a Banach space and the linear map

$$L^p(0,T; W_0^{m,q}(\Omega)) \mapsto X, \quad \xi \mapsto \mathcal{U}\xi \tag{10.20}$$

is an isomorphism.

We write $C([0,T]; W_0^{m,q}(\Omega_t))$ to denote the closed subspace of $L^\infty(0,T; W_0^{m,q}(\Omega_t))$ constituted by functions $\widehat{u}$ such that the corresponding ξ given (10.18) belongs to $C([0,T]; W_0^{m,q}(\Omega))$.

The dual space of $X = L^p(0,T; H_0^1(\Omega_t))$ $\left(1 \le p < \infty, \frac{1}{p} + \frac{1}{p'} = 1\right)$ will be identified with $L^{p'}(0,T; H^{-1}(\Omega_t))$. In what follows we characterize the vectors of this space. In fact, we have by the properties of $\mathcal{U}$ defined in (10.20), that if $S \in X'$ then there exists a unique $R \in L^{p'}(0,T; H^{-1}(\Omega))$ such that

$$\langle S, \widehat{u}\rangle = \langle R, \xi\rangle, \quad \xi = \mathcal{U}^{-1}\widehat{u}.$$

and

$$C_3\|R\| \leq \|S\| \leq C_4\|R\|.$$

To show that, it is sufficient to take $R = \mathcal{U}^* S$ where $\mathcal{U}^*$ is the adjoint operator of $\mathcal{U}$. On the other side, with R we define the operator P :

$$\langle P(t), \widehat{\alpha}\rangle = \langle R(t), \beta\rangle, \quad \widehat{\alpha} \in H_0^1(\Omega_t).$$

where $\beta(y) = k^n(t)\widehat{\alpha}(k(t)y)$. Then

$$C_3\|R(t)\|_{H^{-1}(\Omega)} \leq \|P(t)\|_{H^{-1}(\Omega_t)} \leq C_4\|R(t)\|_{H^{-1}(\Omega)}$$

since

$$C_3\|\beta\|_{H_0^1(\Omega)} \leq \|\widehat{\alpha}\|_{H_0^1(\Omega_t)} \leq C_4\|\beta\|_{H_0^1(\Omega)}.$$

Thus, by identifying S with R and R with P, we obtain that the space $L^{p'}(0,T;H^{-1}(\Omega_t))$ is constituted by the functionals $\widehat{S}$ such that

$$\widehat{S} :]0,T[\to H^{-1}(\Omega_t)), \ \widehat{S} \text{ measurable}$$

$$\begin{aligned}&\exists R \in L^{p'}(0,T;H^{-1}(\Omega)) \text{ satisfying } \langle \widehat{S}(t), \widehat{\alpha}\rangle = \langle R(t), \beta\rangle,\\ &\text{a.e. } t \text{ in }]0,T], \ \beta(y) = k^n(t)\widehat{\alpha}(k(t)y)\end{aligned}$$

and the norm is given by

$$\begin{aligned}\|\widehat{S}\|_{L^{p'}(0,T;H^{-1}(\Omega_t))} &= \left(\int_0^T \|\widehat{S}(t)\|^{p'}_{H^{-1}(\Omega_t)}dt\right)^{\frac{1}{p'}}, \ 1 < p' < \infty\\ \|\widehat{S}\|_{L^\infty(0,T;H^{-1}(\Omega_t))} &= \operatorname*{ess\,sup}_{t\in]0,T[} \|\widehat{S}(t)\|_{H^{-1}(\Omega_t)}.\end{aligned}$$

The space $C([0,T];H^{-1}(\Omega_t))$ will be defined as the closed subspace of $L^\infty(0,T;H^{-1}(\Omega_t))$ constituted by the functionals $\widehat{S}$ such that its corresponding R belongs to $C([0,T];H^{-1}(\Omega))$.

Let $\widehat{u} : \widehat{Q} \to \mathbb{R}$ be a function and

$$\widehat{u}(x,t) = u\left(\frac{x}{k(t)}, t\right), \quad u : Q \to \mathbb{R}$$

then

$$\widehat{u}'(x,t) = -\frac{k'(t)}{k(t)} y_i \frac{\partial u}{\partial y_i}\left(\frac{x}{k(t)}, t\right) + u'\left(\frac{x}{k(t)}, t\right). \tag{10.21}$$

Let $\widehat{u} \in L^p(0,T;L^2(\Omega_t))$, $1 \leq p \leq \infty$, be such that ξ' belongs to $L^p(0,T;H^{-1}(\Omega))$, where $\mathcal{U}\xi = \widehat{u}$. Let $u = k^{-n}\xi$, that is,

$$\widehat{u}(x,t) = k^{-n}(t)\xi\left(\frac{x}{k(t)}, t\right) = u\left(\frac{x}{k(t)}, t\right).$$

Then $u \in L^p(0,T;L^2(\Omega))$ and $u' \in L^p(0,T;H^{-1}(\Omega))$. By (10.21) we have

$$\langle \widehat{u}'(t), \widehat{\alpha} \rangle = \left\langle -\frac{k'(t)}{k(t)} y_i \frac{\partial u}{\partial y_i} + u', \beta \right\rangle$$

where $\widehat{\alpha} \in H_0^1(\Omega_t)$ and $\beta(y) = k^n(t)\widehat{\alpha}(k(t)y)$. Clearly, $\widehat{u}' \in L^p(0,T;H^{-1}(\Omega_t))$.
In particular, if $\widehat{u} \in L^p(0,T;H_0^1(\Omega_t))$ and $u' \in L^p(0,T;H^{-1}(\Omega))$ then

$$(\widehat{u}'(t), \widehat{\alpha})_{L^2(\Omega_t)} = \left(-\frac{k'(t)}{k(t)} y_i \frac{\partial u}{\partial y_i} + u', \beta \right)_{L^2(\Omega)}$$

with $\widehat{\alpha} \in L^2(\Omega_t)$. Clearly $\widehat{u}' \in L^p(0,T;L^2(\Omega_t))$.

We denote by $L^2(0,T;L^2(\Gamma_t))$ the Hilbert space of function

$$\widehat{v} : \widehat{\Sigma} \to \mathbb{R}$$

such that there exists $g \in L^2(0,T;L^2(\Gamma))$ verifying

$$\widehat{v}(x,t) = k^{-n-1}(t) g\left(\frac{x}{k(t)}, t\right),$$

equipped with the inner product

$$(\widehat{v}, \widehat{w})_{L^2(0,T;L^2(\Gamma_t))} = \int_0^T (\widehat{v}(t), \widehat{w}(t))_{L^2(\Gamma_t)} dt.$$

Remark 10.5 *The unit normal vector $\widehat{\eta}(x,t)$ at $(x,t) \in \widehat{\Sigma}$, directed towards the exterior of $\widehat{Q}$, has the form*

$$\widehat{\eta}(x,t) = \{\nu(y), -k'(t)(y,\nu(y))\}[1 + k'^2(t)|(y,\nu(y))|^2]^{-\frac{1}{2}}, \ y = \frac{x}{k(t)}.$$

In fact, fix $(x,t) \in \widehat{\Sigma}$. Let $\varphi = 0$ be a parametrization of a part U of Γ, U containing $y = \frac{x}{k(t)}$. Then a parametrization of a part $\widehat{V}$ of $\widehat{\Sigma}$, $(x,t) \in \widehat{V}$, is $\widehat{\psi}(x,t) = \varphi(\frac{x}{k(t)}) = 0$. We have

$$\nabla \widehat{\psi}(x,t) = \frac{1}{k(t)} \{\nabla \varphi(y), -k'(t)(y, \nabla \varphi(y))\}.$$

From this and observing that $\nu(y) = \frac{\nabla \varphi(y)}{|\nabla \varphi(y)|}$, the remark follows.

Let $\nu^*(x,t)$ be the x-component of $\widehat{\eta}(x,t)$, $|\nu^*(x,t)| = 1$. Then by Remark 10.5, one has

$$\nu^*(x,t) = \nu\left(\frac{x}{k(t)}\right). \tag{10.22}$$

10.5 Proof of the Main Result

10.5.1 Weak Solutions and Solutions by Transposition.

In order to motivate the definition of weak solutions and solutions defined by transposition of the wave equation in $\widehat{Q}$, we obtain some relations between functions. We consider

$$\widehat{u}(x,t) = u\left(\frac{x}{k(t)}, t\right), \quad \widehat{\theta}(x,t) = k^{-n}(t) z\left(\frac{x}{k(t)}, t\right)$$

$$\widehat{v}(x,t) = k^{-n-1}(t) g\left(\frac{x}{k(t)}, t\right), \quad \widehat{v} : \widehat{\Sigma} \to \mathbb{R}$$

One has

$$\widehat{u}'(x,t) = -\frac{k'(t)}{k(t)} y_i \frac{\partial u}{\partial y_i}\left(\frac{x}{k(t)}, t\right) + u'\left(\frac{x}{k(t)}, t\right) \tag{10.23}$$

$$\begin{aligned} \widehat{\theta}'(x,t) = &-nk^{-n-1}(t)k'(t) z\left(\frac{x}{k(t)}, t\right) - \\ &- k^{-n-1}(t)k'(t) y_i \frac{\partial z}{\partial y_i}\left(\frac{x}{k(t)}, t\right) + k^{-n}(t) z'\left(\frac{x}{k(t)}, t\right) \end{aligned} \tag{10.24}$$

and

$$\widehat{u}''(x,t) - \triangle \widehat{u}(x,t) = Lu\left(\frac{x}{k(t)}, t\right),$$

$$\widehat{\theta}''(x,t) - \triangle \widehat{\theta}(x,t) = k^{-n}(t) L^* z\left(\frac{x}{k(t)}, t\right)$$

where L and L^* were defined, respectively, in (10.4) and (10.5).

With the above functions we obtain formally the following results: The change of variable $x = k(t)y$ gives

$$\int_0^T \int_{\Omega_t} (\widehat{u}'' - \triangle \widehat{u}) \widehat{\theta} dx dt = \int_0^T \int_\Omega Lwz dy dt \tag{10.25}$$

$$\int_0^T \int_\Omega wL^* z dy dt = \int_0^T \int_{\Omega_t} \widehat{u}(\widehat{\theta}'' - \triangle \widehat{\theta}) dx dt \tag{10.26}$$

and by (10.22),

$$\begin{aligned} &\int_0^T \int_\Gamma (\delta_{ij} - k'^2 y_i y_j) k^{-2} \frac{\partial z}{\partial y_j} \nu_i g d\Gamma dt = \\ &= \int_0^T \int_{\Gamma_t} (\delta_{ij} - k'^2 k^{-2} x_i x_j) k^{n+1} \frac{\partial \widehat{\theta}}{\partial x_j} \nu_i^* \widehat{v} d\Gamma dt. \end{aligned} \tag{10.27}$$

The Green's formula, the condition $z(t) = 0$ on Γ, the change of variable $x = k(t)y$ and the relations (10.23), (10.24) furnish the identity

$$\begin{aligned} &\int_\Omega [u'(t)z(t) - u(t)z'(t)] dy - \int_\Omega \frac{2k'(t)}{k(t)} y_i \frac{\partial u(t)}{\partial y_i} z(t) dy = \\ &= \int_{\Omega_t} [\widehat{u}'(t)\widehat{\theta}(t) - \widehat{u}(t)\widehat{\theta}'(t)] dx. \end{aligned} \tag{10.28}$$

The Green's formula the integration by parts on $[0,T]$ and the conditions $z(t)=0$ on Γ, $u=g$ on Σ, yield

$$\int_0^T\int_\Omega Lwzdydt = \int_0^T\int_\Omega wL^*zdydt + N(T) - N(0) + J \tag{10.29}$$

where $N(t)$ denotes the left side of (10.28) and J, the left side of (10.27). Then from (10.25)-(10.29) we have

$$\begin{aligned}\int_0^T\int_{\Omega_t}(\widehat{u}''-\triangle\widehat{u})\widehat{\theta}dxdt &= \int_{\Omega_t}[\widehat{u}'(T)\widehat{\theta}(T)-\widehat{u}(T)\widehat{\theta}'(T)]dx-\\ &-\int_{\Omega_0}[\widehat{u}'(0)\widehat{\theta}(0)-\widehat{u}(0)\widehat{\theta}'(0)]dx+\\ &+\int_0^T\int_{\Gamma_t}(\delta_{ij}-k'^2k^{-2}x_ix_j)k^{n+1}\frac{\partial\widehat{\theta}}{\partial x_j}\nu_i^*\widehat{v}d\Gamma dt+\\ &+\int_0^T\int_{\Omega_t}\widehat{u}(\widehat{\theta}''-\triangle\widehat{\theta})dxdt.\end{aligned} \tag{10.30}$$

Motivated by (10.30), we introduce the following problem

$$\left|\begin{array}{l}\widehat{\theta}''-\triangle\widehat{\theta}=\widehat{h} \text{ in } \widehat{Q},\\ \widehat{\theta}=0 \text{ in } \widehat{\Sigma},\\ \widehat{\theta}(0)=\widehat{\theta}^0,\ \widehat{\theta}'(0)=\widehat{\theta}^1 \text{ in } \Omega_0\end{array}\right. \tag{10.31}$$

with data

$$\widehat{\theta}^0\in H_0^1(\Omega_0),\ \widehat{\theta}^1\in L^2(\Omega_0),\ \widehat{h}\in L^1(0,T;L^2(\Omega_t)). \tag{10.32}$$

We say that $\widehat{\theta}$ is a weak solution of Problem (10.31) if

$$\widehat{\theta}\in C([0,T];H_0^1(\Omega_t)),\ \ \widehat{\theta}'\in C([0,T];L^2(\Omega_t))$$

and verifies

$$\left|\begin{array}{l}-\displaystyle\int_0^T(\widehat{\theta}',\widehat{\alpha})_{L^2(\Omega_t)}dt+\int_0^T((\widehat{\theta},\widehat{\alpha}))_{H_0^1(\Omega_t)}dt=\int_0^T(\widehat{h},\widehat{\alpha})_{L^2(\Omega_t)}dt,\\ \forall\widehat{\alpha}\in L^2(0,T;H_0^1(\Omega_t)),\ \widehat{\alpha}'\in L^2(0,T;L^2(\Omega_t)),\\ \widehat{\alpha}(0)=\widehat{\alpha}(T)=0,\ \widehat{\theta}(0)=\widehat{\theta}^0,\ \widehat{\theta}'(0)=\widehat{\theta}^1.\end{array}\right.$$

Theorem 10.5 *Let $\widehat{\theta}(x,t)=k^{-n}(t)z\left(\frac{x}{k(t)},t\right)$. We have that if z is a weak solution of (10.6) then $\widehat{\theta}$ is a weak solution of Problem (10.31) and reciprocally. The data $\{\widehat{\theta}^0,\widehat{\theta}^1,\widehat{h}\}$ and $\{z^0,z^1,h\}$ are related by*

$$\widehat{\theta}^0(x)=k^{-n}(0)z^0\left(\frac{x}{k(0)}\right) \tag{10.33}$$

$$\widehat{\theta}^1(x) = -nk^{-n-1}(0)k'(0)z^0\left(\frac{x}{k(0)}\right) - \\ - k^{n-1}(0)k'(0)y_i\frac{\partial z^0}{\partial y_i}\left(\frac{x}{k(0)}\right) + k^{-n}(0)z^1\left(\frac{x}{k(0)}\right) \tag{10.34}$$

(see (10.23), (10.24)).

Theorem 10.5 is showed by relating integrals on Ω_t and Ω and using Theorem 10.2 and (10.24).

The uniqueness of solutions of Problem (10.31) is a consequence of Theorem 10.5. We also have that, since $\frac{\partial\widehat{\theta}}{\partial x_j} = k^{-n-1}\frac{\partial z}{\partial y_j}$,

$$\frac{\partial\widehat{\theta}}{\partial x_j}, \frac{\partial\widehat{\theta}}{\partial\nu^*} \in L^2(0,T;L^2(\Gamma_t)), \\ \frac{\partial z}{\partial\nu}(y,t) = k^{n+1}(t)\frac{\partial\widehat{\theta}}{\partial\nu^*}(k(t)y,t). \tag{10.35}$$

Remark 10.6 *Clearly we can change the data at time $t = 0$ by final data at $t = T$ in Problem (10.31) and obtain all the above results for the solution $\widehat{\theta}$ of the respective backward problem. In the sequel we introduce the solutions defined by transposition. Let us consider the problem*

$$\left| \begin{array}{l} \widehat{u}'' - \triangle\widehat{u} = 0 \text{ in } \widehat{Q}, \\ \widehat{u} = \widehat{v} \text{ in } \widehat{\Sigma}, \\ \widehat{u}(0) = \widehat{u}^0, \ \widehat{u}'(0) = \widehat{u}^1 \text{ in } \Omega_0 \end{array} \right. \tag{10.36}$$

with data

$$\widehat{u}^0 \in L^2(\Omega_0), \ \widehat{u}^1 \in H^{-1}(\Omega_0), \ \ \widehat{v} \in L^2(0,T;L^2(\Gamma_t)). \tag{10.37}$$

Motivated by (10.30) one introduces the following definition: We say that $\widehat{u} \in L^\infty(0,T;L^2(\Omega_t))$ is a **solution defined by transposition** of Problem (10.36) if $\widehat{u}$ verifies

$$\int_0^T (\widehat{u},\widehat{h})_{L^2(\Omega_t)}dt = \langle\widehat{u}^1,\widehat{\theta}(0)\rangle - \langle\widehat{u}^0,\widehat{\theta}'(0)\rangle_{L^2(\Omega_0)} - \\ - \int_0^T\int_{\Gamma_t}(\delta_{ij} - {k'}^2k^2x_ix_j)k^{n+1}\frac{\partial\widehat{\theta}}{\partial x_j}\nu_i^*\widehat{v}d\Gamma dt, \\ \forall\widehat{h} \in L^1(0,T;L^2(\Omega_t)),$$

(ν^* defined in (10.21) where $\widehat{\theta}$ is the weak solution of the problem

$$\left| \begin{array}{l} \widehat{\theta}'' - \triangle\widehat{\theta} = \widehat{h} \text{ in } \widehat{Q}, \\ \widehat{\theta} = 0 \text{ in } \widehat{\Sigma}, \\ \widehat{\theta}(0) = 0, \ \widehat{\theta}'(0) = 0 \text{ in } \Omega_0. \end{array} \right.$$

Theorem 10.6 *Let* $\widehat{u}(x,t) = u\left(\frac{x}{k(t)}, t\right)$. *We have that if* u *is a solution by transposition of Problem (10.10) then* $\widehat{u}$ *is a solution by transposition of Problem (10.36) and reciprocally. The data* $\{\widehat{u}^0, \widehat{u}^1, \widehat{v}\}$ *and* $\{u^0, u^1, g\}$ *are related by*

$$\widehat{u}^0(x) = u^0\left(\frac{x}{k(0)}\right) \tag{10.38}$$

$$\begin{aligned} &\langle \widehat{u}^1, \widehat{\alpha}\rangle = \left\langle -\frac{k'(0)}{k(0)} y_i \frac{\partial u^0}{\partial y_i} + u^1, \beta \right\rangle, \ \widehat{\alpha} \in H_0^1(\Omega_0), \\ &\widehat{\alpha}(x) = k^{-n}(0)\beta\left(\frac{x}{k(0)}\right) \end{aligned} \tag{10.39}$$

$$\widehat{v}(x,t) = k^{-n-1}(t) g\left(\frac{x}{k(t)}, t\right) \tag{10.40}$$

The proof of Theorem 10.6 is obtained by the same argument used in the proof of (10.30). For the initial conditions one uses the following result:

Remark 10.7 *Let* $\widehat{u}^0 \in L^2(\Omega_t)$ *and* $u^0(y) = \widehat{u}^0(k(t)y)$. *Then*

$$\left\langle x_i \frac{\partial \widehat{u}^0}{\partial x_i}, \widehat{\alpha} \right\rangle = \left\langle y_i \frac{\partial u^0}{\partial y_i}, \beta \right\rangle, \ \widehat{\alpha} \in H_0^1(\Omega_t),\ \widehat{\alpha}(x) = k^{-n}(t)\beta\left(\frac{x}{k(t)}\right).$$

To see this it is enough to make the respective integrations.

From Theorem 10.6 the uniqueness of solution of Problem (10.36) follows and by Theorem 10.3,

$$\widehat{u} \in C([0,T]; L^2(\Omega_t)) \cap C^1([0,T]; H^{-1}(\Omega_t)).$$

We observe that, in addition to (10.26), we have

$$\begin{aligned} &\int_0^T \int_{\Gamma_t} (\delta_{ij} - k'^2 k^{-2} x_i x_j) k^{n+1} \frac{\partial \widehat{\theta}}{\partial x_j} \nu_i^* \nu \frac{\partial \widehat{\theta}}{\partial \nu^*} d\Gamma dt = \\ &= \int_0^T \int_{\Gamma} (\delta_{ij} - k'^2 y_i y_j) k^{-2} \frac{\partial z}{\partial y_j} \nu_i \frac{\partial z}{\partial \nu} d\Gamma dt. \end{aligned}$$

10.5.2 Proof of Theorem 10.1.

Let us consider the system (10.3), that is,

$$\left| \begin{aligned} &\widehat{u}'' - \triangle \widehat{u} = 0 \text{ in } \widehat{Q}, \\ &\widehat{u} = \begin{cases} \widehat{v} & \text{on } \widehat{\Sigma}(y^0), \\ 0 & \text{on } \widehat{\Sigma} \setminus \widehat{\Sigma}(y^0), \end{cases} \\ &\widehat{u}(0) = \widehat{u}^0,\ \widehat{u}'(0) = \widehat{u}^1 \text{ in } \Omega_0 \end{aligned} \right. \tag{10.41}$$

where $\widehat{Q}$ is constructed with $T > T_0$, T_0 given (10.12). With (10.32)- (10.34) and (10.37)-(10.39), we determine, respectively, the isomorphisms

$$G_1\{z^0, z^1\} = \{\widehat{\theta}^0, \widehat{\theta}^1\} \text{ and } G_2\{u^0, u^1\} = \{\widehat{u}^0, \widehat{u}^1\}$$

Consider the operators

$$\sigma\{u^0, u^1\} = \left\{u^1 - \frac{2k'(0)}{k(0)} y_i \frac{\partial u^0}{\partial y_i}, -u^0\right\},$$

$$\Lambda\{z^0, z^1\} = \left\{u'(0) - \frac{2k'(0)}{k(0)} y_i \frac{\partial u(0)}{\partial y_i}, -u(0)\right\},$$

where Λ is the isomorphism defined in (10.16), that is, z is the weak solution of the problem

$$\left| \begin{array}{l} L^* z = 0 \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \\ z(0) = z^0, \ z'(0) = z^1 \text{ in } \Omega \end{array} \right. \tag{10.42}$$

and u the solution defined by transposition of the problem

$$\left| \begin{array}{l} Lu = 0 \text{ in } Q, \\ u = \begin{cases} \dfrac{\partial z}{\partial \nu} & \text{on } \widehat{\Sigma}(y^0), \\ 0 & \text{on } \Sigma \setminus \Sigma(y^0), \end{cases} \\ u(T) = 0, \ u'(T) = 0 \text{ in } \Omega. \end{array} \right. \tag{10.43}$$

Since Λ is an isomorphism we have that for each $\{u^1, u^0\} \in H^{-1}(\Omega) \times L^2(\Omega)$ there exists an unique $\{z^0, z^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\Lambda\{z^0, z^1\} = \left\{u^1 - \frac{2k'(0)}{k(0)} y_i \frac{\partial u^0}{\partial y_i}, -u^0\right\}. \tag{10.44}$$

Thus, if u is the solution of problem (10.43) constructed with $\{z^0, z^1\}$, we have

$$u(0) = u^0, \quad u'(0) = u^1.$$

With the above operators we determine the isomorphism

$$\Lambda_1 = G_1 \Lambda^{-1} \sigma G_2^{-1}, \text{ that is}$$

$$\begin{array}{rccc} \Lambda_1 : & L^2(\Omega_0) \times H^{-1}(\Omega_0) & \to & H_0^1(\Omega_0) \times L^2(\Omega_0) \\ & \{\widehat{u}^0, \widehat{u}^1\} & \mapsto & \Lambda_1\{\widehat{u}^0, \widehat{u}^1\} = \{\widehat{\theta}^0, \widehat{\theta}^1\} \end{array} \tag{10.45}$$

Let $\{\widehat{u}^0, \widehat{u}^1\} \in L^2(\Omega_0) \times H^{-1}(\Omega_0)$. Thus, by (10.45), we determine $\{\widehat{\theta}^0, \widehat{\theta}^1\}$. With this data we find the weak solution $\widehat{\theta}$ of the problem

$$\left| \begin{array}{l} \widehat{\theta}'' - \triangle \widehat{\theta} = 0 \text{ in } \widehat{Q}, \\ \widehat{\theta} = 0 \text{ in } \widehat{\Sigma}, \\ \widehat{\theta}(0) = \widehat{\theta}^0, \ \widehat{\theta}'(0) = \widehat{\theta}^1 \text{ in } \Omega_0 \end{array} \right. \tag{10.46}$$

and with $\{z^0, z^1\} = G_1^{-1}\{\widehat{\theta}^0, \widehat{\theta}^1\}$, the weak solution z of the problem

$$\left| \begin{array}{l} L^* z = 0 \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \\ z(0) = z^0, \ z'(0) = z^1 \text{ in } \Omega. \end{array} \right.$$

Next, we determine the solution defined by transposition $\widetilde{u}$ of the problem

$$\left| \begin{array}{l} L\widetilde{u} = 0 \text{ in } Q, \\ \widetilde{u} = \begin{cases} \dfrac{\partial z}{\partial \nu} & \text{on } \widehat{\Sigma}(y^0), \\ 0 & \text{on } \Sigma \setminus \Sigma(y^0), \end{cases} \\ \widetilde{u}(0) = u^0, \ \widetilde{u}'(0) = u^1 \text{ in } \Omega \end{array} \right. \tag{10.47}$$

where $\{u^0, u^1\}$ and $\{z^0, z^1\}$ are related by (10.44). We have by the uniqueness of solutions of problem (10.47) that $\widetilde{u} = u$, u the solution of (10.43) constructed with $\{z^0, z^1\}$. Therefore

$$\widetilde{u}(T) = 0, \quad \widetilde{u}'(T) = 0.$$

Finally, from Theorem 10.6, it follows that $\widehat{u}(x,t) = \widetilde{u}\left(\frac{x}{k(t)}, t\right)$ is the solution defined by transposition of Problem (10.41) and $\widehat{u}$ satisfies the final condition

$$\widehat{u}(T) = 0, \quad \widehat{u}'(T) = 0.$$

By (10.35) and (10.36), we have that the control $\widehat{v}$ has the form

$$\widehat{v} = \frac{\partial \widehat{\theta}}{\partial \nu^*}, \quad \widehat{\theta} \text{ weak solution (10.46).}$$

Thus, the proof of Theorem 10.1 is concluded.

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